22 RANDOM SIMPLICIAL COMPLEXES
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INTRODUCTION

Random shapes arise naturally in many contexts. The topological and geometric structure of such objects is interesting for its own sake, and also for applications. In physics, for example, such objects arise naturally in quantum gravity, in material science, and in other settings. Stochastic topology may also be considered as a null hypothesis for topological data analysis.

In this chapter we overview combinatorial aspects of stochastic topology. We focus on the topological and geometric properties of random simplicial complexes. We introduce a few of the fundamental models in Section 22.1. We review high-dimensional expander-like properties of random complexes in Section 22.2. We discuss threshold behavior and phase transitions in Section 22.3, and Betti numbers and persistent homology in Section 22.4.

22.1 MODELS

We briefly introduce a few of the most commonly studied models.

22.1.1 ERDŐS–RÉNYI-INSPIRED MODELS

A few of the models that have been studied are high-dimensional analogues of the Erdős–Rényi random graph.

The Erdős–Rényi random graph

The Erdős–Rényi random graph $G(n, p)$ is the probability distribution on all graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where every edge is included with probability p jointly independently. Standard references include [Bol01] and [JŁR00].

One often thinks of p as a function of n and studies the asymptotic properties of $G(n, p)$ as $n \to \infty$. We say that an event happens *with high probability (w.h.p.)* if the probability approaches 1 as $n \to \infty$.

Erdős–Rényi showed that $\bar{p} = \log n / n$ is a sharp threshold for connectivity. In other words,: for every fixed $\epsilon > 0$, if $p \geq (1 + \epsilon)\bar{p}$ then w.h.p. $G(n, p)$ is connected, and if $p \leq (1 - \epsilon)\bar{p}$ then w.h.p. it is disconnected. A slightly sharper statement is given in the following section. Several thresholds for topological properties of $G(n, p)$ are summarized in Table 22.1.1.

TABLE 22.1.1 Topological thresholds for $G = G(n,p)$. The column SHARP indicates whether the threshold is sharp, coarse, or one-sided sharp. The column TIGHT indicates whether there is any room for improvement on the present bound.

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
G is not 0-collapsible	$1/n$	one-sided	yes	[Pit88]
$H_1(G) \neq 0$	$1/n$	one-sided	yes	[Pit88]
G is not planar	$1/n$	sharp	yes	[ŁPW94]
G contains arbitrary minors	$1/n$	sharp	yes	[AKS79]
G is pure 1-dimensional	$\log n/n$	sharp	yes	[ER59]
G is connected	$\log n/n$	sharp	yes	[ER59]

GLOSSARY

Threshold function: Let $\mathcal{P}$ be a graph property. We say that f is a threshold function for property $\mathcal{P}$ in the random graph $G \in G(n,p)$ if whenever $p = \omega(f)$, G has property $\mathcal{P}$ w.h.p. and whenever $p = o(f)$, G does not have property $\mathcal{P}$.

Sharp threshold: We say that f is a sharp threshold for graph property $\mathcal{P}$ if there exists a function $g = o(f)$ such that for $p < f - g$, $G \notin \mathcal{P}$ w.h.p. and if $p > f + g$, $G \in \mathcal{P}$ w.h.p.

Simplicial complex: A simplicial complex Δ is a collection of subsets of a set S, such that (1) if $U \subset V$ is nonempty and $V \in \Delta$ then $U \in \Delta$, and (2) $\{v\} \in \Delta$ for every $v \in S$. An element of Δ is called a face. Such a set system can be naturally associated a topological space by considering every set of size k in Δ to represent a $(k-1)$-dimensional simplex, homeomorphic to a closed Euclidean ball. This topological space is sometimes called the geometric realization of Δ, but we will slightly abuse notation and identify a simplicial complex with its geometric realization.

Link: Given a simplicial complex Δ and a face $\sigma \in \Delta$, the link of σ in Δ is defined by

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}.$$

The link is itself a simplicial complex.

Homology: Associated with any simplicial complex X, abelian group G, and integer $i \geq 0$, $H_i(X,G)$ denotes the ith homology group of X with coefficients in G. If k is a field, then $H_i(X,k)$ is a vector space over k.

Homology is defined as "cycles modulo boundaries." Homology is invariant under homotopy deformations.

Betti numbers: If one considers homology with coefficients in $\mathbb{R}$, then $H_i(X,\mathbb{R})$ is a real vector space. The Betti numbers β_i are defined by $\beta_i = \dim H_i(X,\mathbb{R})$. The 0th Betti number β_0 counts the number of connected components of X, and in general the ith Betti number is said to count the number of i-dimensional holes in X.

The random 2-complex

Random hypergraphs have been well studied, but if we wish to study such objects topologically then random simplicial complexes is probably a more natural point of view.

Linial and Meshulam introduced the topological study of the random 2-complex $Y(n,p)$ in [LM06]. This model of random simplicial complex has n vertices, $\binom{n}{2}$ edges, and each of the $\binom{n}{3}$ possible 2-dimensional faces is included independently with probability p.

The random 2-complex is perhaps the most natural 2-dimensional analogue of $G(n,p)$. For example, the link of every vertex in $Y(n,p)$ has the same distribution as $G(n-1,p)$.

Several topological thresholds for $Y(n,p)$ discussed in the next section are described in Table 22.1.2.

TABLE 22.1.2   Topological thresholds for the random 2-complex $Y = Y(n,p)$. c.f. in the TIGHT column means that the bound is best possible up to a constant factor.

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
Y is not 1-collapsible	$2.455/n$	one-sided	yes	[CCFK12, ALŁM13, AL16]
$H_2(Y,\mathbb{R}) \neq 0$	$2.753/n$	one-sided	yes	[Koz10, LP14, AL15]
cdim $\pi_1(Y) = 2$	$\Theta(1/n)$	?	c.f.	[CF15a, New16]
Y is not embeddable in $\mathbb{R}^4$	$\Theta(1/n)$	?	c.f.	[Wag11]
Y is pure 2-dimensional	$2\log n/n$	sharp	yes	[LM06]
$H_1(Y,\mathbb{Z}/\ell\mathbb{Z}) = 0$	$2\log n/n$	sharp	yes	[LM06, MW09]
$H_1(Y,\mathbb{R}) = 0$	$2\log n/n$	sharp	yes	[LM06, HKP12]
$\pi_1(Y)$ has property (T)	$2\log n/n$	sharp	yes	[HKP12]
$H_1(Y,\mathbb{Z}) = 0$	$O(\log n/n)$	sharp	c.f.	[HKP17]
cdim $\pi_1(Y) = \infty$	$1/n^{3/5}$	coarse	yes	[CF13]
Y contains arbitrary subdivisions	$\theta\left(1/\sqrt{n}\right)$	?	c.f.	[GW16]
$\pi_1(Y) = 0$	$O(1/\sqrt{n})$	?	no	[BHK11, GW16, KPS16]

TABLE 22.1.3   Topological thresholds for $Y_d(n,p)$, $d > 2$. Definitions for the constants c_d and c_d^* are given in Section 22.3.2.

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
Y is not $(d-1)$-collapsible	c_d/n	one-sided	yes	[ALŁM13, AL16]
$H_d(Y,\mathbb{R}) \neq 0$	c_d^*/n	one-sided	yes	[Koz10, LP14, AL15]
Y is not embeddable in $\mathbb{R}^{2d}$	$\Theta(1/n)$	?	c.f.	[Wag11]
Y is pure d-dimensional	$d\log n/n$	sharp	yes	[MW09]
$H_{d-1}(Y,\mathbb{Z}/\ell\mathbb{Z}) = 0$	$d\log n/n$	sharp	yes	[MW09, HKP12]
$H_{d-1}(Y,\mathbb{R}) = 0$	$d\log n/n$	sharp	yes	[HKP12]
$H_{d-1}(Y,\mathbb{Z}) = 0$	$O(\log n/n)$	sharp	c.f.	[HKP17]
$\pi_{d-1}(Y) = 0$	$O(\log n/n)$	sharp	c.f.	[HKP17]

The random d-complex

The natural generalization to d-dimensional model was introduced by Meshulam and Wallach in [MW09]. For the random d-complex $Y_d(n, p)$, contains the complete $(d-1)$-skeleton of a simplex on n vertices, and every d-dimensional face appears independently with probability p. Some of the topological subtlety of the random 2-dimensional model collapses in higher dimensions: for $d \geq 3$, the complexes are $d-2$-connected, and in particular simply connected. By the Hurewicz theorem, $\pi_{d-1}(Y)$ is isomorphic to $H_{d-1}(Y, \mathbb{Z})$, so these groups have the same vanishing threshold.

The random clique complex

Another analogue of $G(n, p)$ in higher dimensions was introduced in [Kah09]. The random clique complex $X(n, p)$ is the *clique complex* of $G(n, p)$. It is the maximal simplicial complex compatible with a given graph. In other words, the faces of the clique complex $X(H)$ correspond to complete subgraphs of the graph H.

The random clique complex asymptotically puts a measure over a wide range of topologies. Indeed, every simplicial complex is homeomorphic to a clique complex, e.g., by barycentric subdivision.

There are several comparisons of this model to the random d-complex, but some important contrasts as well. One contrast to $Y_d(n, p)$ is that for every $k \geq 1$, $X(n, p)$ has not one but two phase transitions for kth homology, one where homology appears and one where it vanishes. In particular, higher homology is not monotone with respect to p. However, there are still comparisons to $Y_d(n, p)$— the appearance of homology $H_k(X)$ is analogous to the birth of top homology $H_d(Y)$. Similarly, the vanishing threshold for $H_k(X(n, p))$ is analogous to vanishing of $H_{d-1}(Y)$.

TABLE 22.1.4   Topological thresholds for $X(n, p)$

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
$\pi_1(X) = 0$	$p = 1/n^{1/3}$	?	no	[Kah09, Bab12, CFH15]
$H_k(X, \mathbb{R}) \neq 0$	$p = c/n^{1/k}$	one-sided	c.f.	[Kah09]
X^k is pure k-dimensional	$p = \left(\frac{(k/2+1)\log n}{n} \right)^{1/(k+1)}$	yes	yes	[Kah09]
$H_k(Y, \mathbb{R}) = 0$	$p = \left(\frac{(k/2+1)\log n}{n} \right)^{1/(k+1)}$	yes	yes	[Kah14a]

The multi-parameter model

There is a natural multi-parameter model which generalizes all of the models discussed so far. For every $i = 1, 2, \ldots$ let $p_i : \mathbb{N} \to [0, 1]$. Then define the multiparameter random complex $X(n; p_1, p_2, \ldots)$ as follows. Start with n vertices. Insert every edge with probability p_1. Conditioned on the presence of all three boundary edges, insert a 2-face with probability p_2, etc.

The random d-complex $Y_d(n, p)$ with complete $(d-1)$-skeleton is equivalent to the case $p_1 = p_2 = \cdots = p_{d-1} = 1$, $p = p_d$, and $p_{d+1} = p_{d+2} = \cdots = 0$. The random clique complex $X(n, p)$ corresponds to $p_2 = p_3 = \cdots = 1$, and $p = p_1$. This multi-parameter model is first studied by Costa and Farber in [CF16].

22.1.2 RANDOM GEOMETRIC MODELS

The *random geometric graph* $\mathcal{G}(n, r)$ is a flexible model, defined as follows. Consider a probability distribution on $\mathbb{R}^d$ with a bounded, measurable, density function $f : \mathbb{R}^d \to \mathbb{R}$. Then one chooses n points independently and identically distributed (i.i.d.) according to this distribution. The n points are the vertices of the graph, and two vertices are adjacent if they are within distance r. Usually $r = r(n)$ and $n \to \infty$. The standard reference for random geometric graphs is Penrose's monograph [Pen03].

There are at least two commonly studied ways to build a simplicial complex on a geometric graph. The first is the Vietoris–Rips complex, which is the same construction as the clique complex above—one fills in all possible faces, i.e., the faces of the Vietoris–Rips complex correspond to the cliques of the graph. The second is the Čech complex, where one considers the higher intersections of the balls of radius $r/2$. This leads to two natural models for random geometric complexes $VR(n, r)$ and $C(n, r)$.

TABLE 22.1.5   Topological thresholds for $C(n, r)$

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
$H_k(X) \neq 0$	$nr^d = n^{-(k+2)/(k+1)}$	coarse	yes	[Kah11]
$H_k(Y) = 0$	$nr^d = \log n + \theta(\log \log n)$	sharp	essentially	[BW17]

TABLE 22.1.6   Topological thresholds for $VR(n, r)$

PROPERTY	THRESHOLD	SHARP	TIGHT	SOURCE
$H_k(X) \neq 0$	$nr^d = n^{-(2k+2)/(2k+1)}$	coarse	yes	[Kah11]
$H_k(Y) = 0$	$nr^d = \theta(\log n)$	sharp	c.f.	[Kah11]

22.2 HIGH-DIMENSIONAL EXPANDERS

GLOSSARY

Cheeger number: The normalized Cheeger number of a graph $h(G)$ with vertex set V is defined by

$$h(G) = \min_{\emptyset \subsetneq A \subsetneq V} \frac{\#E(A, \bar{A})}{\min\{\text{vol } A, \text{vol } \bar{A}\}},$$

where

$$\text{vol } A = \sum_{v \in A} \deg(v)$$

and $\bar{A}$ is the complement of A in V.

Laplacian: For a connected graph H, the normalized graph Laplacian $L = L[H]$ is defined by

$$L = I - D^{-1/2}AD^{-1/2}.$$

Here A is the adjacency matrix, and D is the diagonal matrix with vertex degrees along the diagonal.

Spectral gap: The eigenvalues of the normalized graph Laplacian of a connected graph satisfy

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq 2.$$

The smallest positive eigenvalue $\lambda_2[H]$ is of particular importance, and is sometimes called the spectral gap of H.

Expander family: Let $\{G_i\}$ be an infinite sequence of graphs where the number of vertices tends to infinity. We say that $\{G_i\}$ is an expander family if

$$\liminf \lambda_2[G_i] > 0.$$

Expander graphs are of fundamental importance for their applications in computer science and mathematics [HLW06]. It is natural to seek their various higher-dimensional generalizations. See [Lub14] for a survey of recent progress on higher-dimensional expanders, particularly Ramanujan complexes which generalize Ramanujan graphs.

Gromov suggested that one property that higher-dimensional expanders should have is geometric or topological overlap. A sequence of d-dimensional simplicial complexes Δ_1, Δ_2, ..., is said to have the *geometric overlap property* if for every geometric map (affine-linear on each face) $f : \Delta_i \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ such that $f^{-1}(p)$ intersects the interior of a constant fraction of the d-dimensional faces. The sequence is said to have the stronger *topological overlap property* if this holds even for continuous maps.

One way to define a higher-dimensional expander is via *coboundary expansion*, which generalizes the Cheeger number of a graph. Following Linial and Meshulam's coisoperimetric ideas, Dotterer and Kahle [DK12] pointed out that d-dimensional random simplicial complexes are coboundary expanders. By a theorem of Gromov [Gro09, Gro10] (see the note [DKW15] for a self-contained proof of Gromov's theorem), this implies that random complexes have the topological overlap property.

Lubotzky and Meshulam introduced a new model of random 2-complex, based on random Latin squares, in [LM15]. The main result is the existence of coboundary expanders with bounded edge-degree, answering a question asked implicitly in [Gro10] and explicitly in [DK12].

Another way to define higher-dimensional expanders is via the *spectral gap* of various Laplacian operators. Hoffman, Kahle, and Paquette studied the spectral gap of random graphs [HKP12], and applied Garland's method to prove homology-vanishing theorems. Gundert and Wagner extended this to study higher-order spectral gaps of these complexes [GW16]. Parzanchevski, Rosenthal, and Tessler showed that this implies the geometric overlap property [PRT15].

22.3 PHASE TRANSITIONS

There has been a lot of interest in identifying thresholds for various topological properties, such as vanishing of homology. As some parameter varies, the topology passes a *phase transition* where some property suddenly emerges. In this section we review a few of the most well-studied topological phase transitions.

22.3.1 HOMOLOGY-VANISHING THEOREMS

The following theorem describing the connectivity threshold for the random graph $G(n, p)$ is the archetypal homology-vanishing theorem.

THEOREM 22.3.1   *Erdős–Rényi theorem*   [ER59]

If

$$p \geq \frac{\log n + \omega(1)}{n}$$

then w.h.p. $G(n, p)$ is connected, and if

$$p \leq \frac{\log n - \omega(1)}{n}$$

then w.h.p. $G(n, p)$ is disconnected. Here $\omega(1)$ is any function so that $\omega(1) \to \infty$ as $n \to \infty$.

We say that it is a homology-vanishing theorem because path connectivity of a topological space X is equivalent to $\widetilde{H}_0(X, G) = 0$ with any coefficient group G.

It is also a cohomology-vanishing theorem, since $\widetilde{H}^0(X, G) = 0$ is also equivalent to path-connectivity. In many ways it is better to think of it as a cohomology theorem, since the standard proof, for example in [Bol01][Chapter 10], is really a cohomological one. This perspective helps when understanding the proof of the Linial–Meshulam theorem.

The following cohomological analogue of the Erdős–Rényi theorem was the first nontrivial result for the topology of random simplicial complexes.

THEOREM 22.3.2   *Linial–Meshulam theorem*   [LM06]

Let $Y = Y(n, p)$. If

$$p \geq \frac{2 \log n + \omega(1)}{n}$$

then w.h.p. $H_1(Y, \mathbb{Z}/2\mathbb{Z}) = 0$, and if

$$p \leq \frac{2 \log n - \omega(1)}{n}$$

then w.h.p. $H_1(Y, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

One of the main tools introduced in [LM06] is a new co-isoperimetric inequality for the simplex, which was discovered independently by Gromov. These co-isoperimetric inequalities were combined by Linial and Meshulam with intricate cocycle-counting combinatorics to get a sharp threshold.

See [DKW15] for a comparison of various definitions co-isoperimetry, and a clean statement and self-contained proof of Gromov's theorem.

Theorem 22.3.2 was generalized further by Meshulam and Wallach.

THEOREM 22.3.3 [MW09]

Fix $d \geq 1$, and let $Y = Y_d(n, p)$. Let G be any finite abelian group. If

$$p \geq \frac{d \log n + \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, G) = 0$, and if

$$p \leq \frac{d \log n - \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, G) \neq 0$.

Theorem 22.3.3 generalizes Theorem 22.3.2 in two ways: by letting the dimension $d \geq 2$ be arbitrary, and also by letting coefficients be in an arbitrary finite abelian group G.

Spectral gaps and Garland's method

There is another approach to homology-vanishing theorems for simplicial complexes, via Garland's method [Gar73]. The following refinement of Garland's theorem is due to Ballman and Świątkowski.

THEOREM 22.3.4 [Gar73, BŚ97]

If Δ is a finite, pure d-dimensional, simplicial complex, such that

$$\lambda_2[lk_\Delta(\sigma)] > 1 - \frac{1}{d}$$

for every $(d-2)$-dimensional face $\sigma \in \Delta$, then $H_{d-1}(\Delta, \mathbb{R}) = 0$.

This leads to a new proof of Theorem 22.3.3, at least over a field of characteristic zero.

THEOREM 22.3.5 [HKP12]

Fix $d \geq 1$, and let $Y = Y_d(n, p)$. If

$$p \geq \frac{d \log n + \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, \mathbb{R}) = 0$, and if

$$p \leq \frac{d \log n - \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, \mathbb{R}) \neq 0$. Here $\omega(1)$ is any function that tends to infinity as $n \to \infty$.

This is slightly weaker than the Meshulam–Wallach theorem topologically speaking, since $H_i(Y, G) = 0$ for any finite group G implies that $H_i(Y, \mathbb{R}) = 0$ by the universal coefficient theorem, but generally the converse is false. However, the

proof via Garland's method avoids some of the combinatorial complications of co-cycle counting.

Garland's method also provides proofs of theorems which have so far eluded other methods. For example, we have the following homology-vanishing threshold in the random clique complex model. Note that $k = 0$ again corresponds to the Erdős–Rényi theorem.

THEOREM 22.3.6 [Kah14a]

Fix $k \geq 1$ and let $X = X(n, p)$. Let $\omega(1)$ denote a function that tends to ∞ arbitrarily slowly. If

$$p \geq \left(\frac{\left(\frac{k}{2} + 1\right) \log n + \left(\frac{k}{2}\right) \log \log n + \omega(1)}{n} \right)^{1/(k+1)}$$

then w.h.p. $H_k(X, \mathbb{R}) = 0$, and if

$$\frac{1}{n^k} \leq p \leq \left(\frac{\left(\frac{k}{2} + 1\right) \log n + \left(\frac{k}{2}\right) \log \log n - \omega(1)}{n} \right)^{1/(k+1)}$$

then w.h.p. $H_k(X, \mathbb{R}) \neq 0$.

It may be that this theorem holds with $\mathbb{R}$ coefficients replaced by a finite group G or even with $\mathbb{Z}$, but for the most part this remains an open problem. The only other case that seems to be known is the case $k = 1$ and $G = \mathbb{Z}/2$ by DeMarco, Hamm, and Kahn [DHK13], where a similarly sharp threshold is obtained.

The applications of Garland's method depends on new results on the spectral gap of random graphs.

THEOREM 22.3.7 [HKP12]

Fix $k \geq 0$. Let $\lambda_1, \lambda_2, \ldots$ denote the eigenvalues of the normalized graph Laplacian of the random graph $G(n, p)$. If

$$p \geq \frac{(k+1) \log n + \omega(1)}{n}$$

then

$$1 - \sqrt{\frac{C}{np}} \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 1 + \sqrt{\frac{C}{np}}$$

with probability at least $1 - o\left(n^{-k}\right)$. Here $C > 0$ is a universal constant.

Theorem 22.3.6, combined with some earlier results [Kah09], has the following corollary.

THEOREM 22.3.8 [Kah14a]

Let $k \geq 3$ and $\epsilon > 0$ be fixed. If

$$\left(\frac{(C_k + \epsilon) \log n}{n} \right)^{1/k} \leq p \leq \frac{1}{n^{1/(k+1)+\epsilon}},$$

where $C_3 = 3$ and $C_k = k/2 + 1$ for $k > 3$, then w.h.p. X is rationally homotopy equivalent to a bouquet of k-dimensional spheres.

The main remaining conjecture for the topology of random clique complexes is that these rational homotopy equivalences are actually homotopy equivalences.

CONJECTURE 22.3.9   *The bouquet-of-spheres conjecture.*

Let $k \geq 3$ and $\epsilon > 0$ be fixed. If

$$\frac{n^\epsilon}{n^{1/k}} \leq p \leq \frac{n^{-\epsilon}}{n^{1/(k+1)}}$$

then w.h.p. X is homotopy equivalent to a bouquet of k-spheres.

Given earlier results, this is equivalent to showing that $H_k(X, \mathbb{Z})$ is torsion free. So far, integer homology for $X(n, p)$ is not very well understood. Some progress has been made for $Y_d(n, p)$, described in the following.

Integer homology

Unfortunately, neither method discussed above (the cocycle-counting methods pioneered by Linial and Meshulam or the spectral methods of Garland), seems to handle integral homology. There is a slight subtlety here—if one knows for some simplicial complex Σ that $H_i(\Sigma, G) = 0$ for every finite abelian group G then $H_i(\Sigma, \mathbb{Z}) = 0$ by the universal coefficient theorem. See, for example, Hatcher [Hat02][Chapter 2].

So it might seem that the Theorem 22.3.3 will also handle $\mathbb{Z}$ coefficients, but the proof uses cocycle counting methods which require G to be fixed, or at least for the order of the coefficient group $|G|$ to be growing sufficiently slowly. Cocycle counting does not seem to work, for example, when $|G|$ is growing exponentially fast. The following gives an upper bound on the vanishing threshold for integer homology.

THEOREM 22.3.10   [HKP12]

Fix $d \geq 2$, and let $Y = Y_d(n, p)$. If

$$p \geq \frac{80d \log n}{n},$$

then w.h.p. $H_{d-1}(Y, \mathbb{Z}) = 0$.

The author suspects that the true threshold for homology with $\mathbb{Z}$ coefficients is the same as for field coefficients: $d \log n / n$.

CONJECTURE 22.3.11   *A sharp threshold for $\mathbb{Z}$ homology.*

If

$$p \geq \frac{d \log n + \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, \mathbb{Z}) = 0$, and if

$$p \leq \frac{d \log n - \omega(1)}{n}$$

then w.h.p. $H_{d-1}(Y, \mathbb{Z}) \neq 0$.

22.3.2 THE BIRTH OF CYCLES AND COLLAPSIBILITY

$G(n, p)$ **in the** $p = 1/n$ **regime**

There is a remarkable phase transition in structure of the random graph $G(n, p)$ at the threshold $p = 1/n$. A "giant" component, on a constant fraction of the vertices, suddenly emerges. This is considered an analogue of percolation on an infinite lattice, where an infinite component appears with probability 1.

THEOREM 22.3.12 [ER59]

Let $p = c/n$ for some $c > 0$ fixed, and $G = G(n, p)$.

- *If $c < 1$ then w.h.p. all components are of order $O(\log n)$.*

- *If $c > 1$ then w.h.p. there is a unique giant component, of order $\Omega(n)$.*

An overview of this remarkable phase transition can be found in Chapter 11 of Alon and Spencer [AS08].

In random graphs, the appearance of cycles with high probability has the same threshold $1/n$.

THEOREM 22.3.13 [Pit88]

Suppose $p = c/n$ where $c > 0$ is constant.

- *If $c \geq 1$ then w.h.p. G contains at least one cycle, i.e.,*

$$\mathbb{P}[H_1(G) \neq 0] \to 1.$$

- *If $c < 1$ then*
$$\mathbb{P}[H_1(G) = 0] \to \sqrt{1-c} \, \exp(c/2 + c^2/4).$$

The analogy in higher dimensions is only just beginning to be understood.

The birth of cycles

Kozlov first studied the vanishing threshold for top homology in [Koz10].

THEOREM 22.3.14 [Koz10]

Let $Y = Y_d(n, p)$, and G be any abelian group.

(1) *If $p = o(1/n)$ then w.h.p. $H_d(Y, G) = 0$.*

(2) *If $p = \omega(1/n)$ then w.h.p. $H_d(Y, G) \neq 0$.*

Part (1) of this theorem cannot be improved. Indeed, let S be the number of subcomplexes isomorphic to the boundary of a $(d + 1)$-dimensional simplex. If $p = c/n$ for some constant $c > 0$, then

$$\mathbb{E}[S] \to c^{d+2}/(d+2)!,$$

as $n \to \infty$. Moreover, S converges in law to a Poisson distribution with this mean in the limit, so

$$\mathbb{P}[H_d(Y, G) \neq 0] \geq \mathbb{P}[S \neq 0] \to 1 - \exp(-c^{d+2}/(d+2)!).$$

In particular, for $p = c/n$ and $c > 0$, $\mathbb{P}[H_d(Y,G) \neq 0]$ is bounded away from zero.

On the other hand, part (2) can be improved. Indeed straightforward computation shows that if $p \geq c/n$ and $c > d+1$ then w.h.p. the number of d-dimensional faces is greater than the number of $(d-1)$-dimensional faces. Simply by dimensional considerations, we conclude that $H_d(Y,G) \neq 0$.

This can be improved more though. Aronshtam and Linial found the best possible constant factor c_d^*, defined for $d \geq 2$ as follows.

Let $x \in (0,1)$ be the unique root to the equation

$$(d+1)(1-x) + (1+dx)\log x = 0,$$

and then set

$$c_d^* = \frac{-\log x}{(1-x)^d}.$$

THEOREM 22.3.15 [AL15]

Let $Y = Y_d(n,p)$. If $p \geq c/n$ where $c > c_d^*$, then w.h.p. $H_d(Y,G) \neq 0$.

In the other direction, Linial and Peled showed that this result is tight, at least in the case of $\mathbb{R}$ coefficients.

THEOREM 22.3.16 [LP14]

If $p \leq c/n$ where $c < c_d^*$ then w.h.p. $H_d(Y,G)$ is generated by simplex boundaries. So
$$\mathbb{P}\left[H_d(Y,\mathbb{R}) = 0\right] \to \exp(-c^{d+2}/(d+2)!).$$

Linial and Peled also showed the birth of a *giant (homological) shadow* at the same point. This is introduced and defined in [LP14], and it is discussed there as a higher-dimensional analogue of the birth of the giant component in $G(n,p)$.

The threshold for d-collapsibility

In a d-dimensional simplicial complex, an *elementary collapse* is an operation that deletes a pair of faces (σ, τ) such that

- τ is a d-dimensional face,

- σ is a $(d-1)$-dimensional face contained in τ, and

- σ is not contained in any other d-dimensional faces.

An elementary collapse results in a homotopy equivalent simplicial complex.

If a simplicial complex can be reduced to a $(d-1)$-dimensional complex by a series of elementary collapses, we say that it is *d-collapsible*.

For a graph, 1-collapsible is equivalent to being a forest. In other words, a graph G is 1-collapsible if and only if $H_1(G) = 0$. This homological criterion does not hold in higher dimensions. In fact, somewhat surprisingly, d-collapsibility and $H_d \neq 0$ have distinct thresholds for random complexes.

Let $d \geq 2$ and set

$$g_d(x) = (d+1)(x+1)e^{-x} + x(1 - e^{-x})^{d+1}.$$

Define c_d to be the unique solution $x > 0$ of $g_d(x) = d + 1$.

THEOREM 22.3.17 [ALŁM13, AL16]

Let $Y = Y_d(n, p)$.

- *If $p \geq c/n$ where $c > c_d$ then w.h.p. Y is not d-collapsible, and*

- *if $p \leq c/n$ where $c < c_d$ then Y is d-collapsible with probability bounded away from zero.*

So again, this is a one-sided sharp threshold. Regarding collapsibility in the random clique complex model, Malen showed in his PhD thesis [Mal16] that if $p \ll n^{-1/(k+1)}$, then w.h.p. $X(n, p)$ is k-collapsible.

Embeddability

Every d-dimensional simplicial complex is embeddable in $\mathbb{R}^{2d+1}$, but not necessarily in $\mathbb{R}^{2d}$. Wagner studied the threshold for non-embeddability of random d-complexes in $\mathbb{R}^{2d}$, and showed the following for $Y = Y_d(n, p)$.

THEOREM 22.3.18 [Wag11]

There exist constants $c_1, c_2 > 0$ depending only on the dimension d such that:

- *if $p < c_1/n$ then w.h.p. Y is embeddable in $\mathbb{R}^{2d}$, and*

- *if $p > c_2/n$ then w.h.p. Y is not embeddable in $\mathbb{R}^{2d}$.*

There is a folklore conjecture that a d-dimensional simplicial complex on n vertices embeddable in $\mathbb{R}^{2d}$ can have at most $O(n^d)$ faces [Dey93, Kal91]. See, for example, the discussion in the expository book chapter [Wag13]. The $d = 1$ case is equivalent to showing that a planar graph may only have linearly many edges, which follows immediately from the Euler formula, but the conjecture is open for every $d \geq 2$. Theorem 22.3.18 shows that it holds generically.

22.3.3 PHASE TRANSITIONS FOR HOMOLOGY IN RANDOM GEOMETRIC COMPLEXES

Penrose described sharp thresholds for connectivity of random geometric graphs, analogous to the Erdős–Rényi theorem. In the case of a uniform distribution on the unit cube $[0, 1]^d$ or a standard multivariate distribution, these results are tight [Pen03].

Thresholds for homology in random geometric complexes was first studied in [Kah11]. A homology vanishing threshold for random geometric complexes is obtained in [Kah11], which is tight up to a constant factor, but recently a much sharper result was obtained by Bobrowski and Weinberger.

THEOREM 22.3.19 [BW17]

Fix $1 \geq k \geq d - 1$. If

$$nr^d \geq \log n + k \log \log n + \omega(1),$$

then w.h.p. $\beta_k = 0$, and if

$$nr^d \leq \log n + (k-2) \log \log n - \omega(\log \log \log n),$$

then w.h.p. $\beta_k \to \infty$.

22.3.4 RANDOM FUNDAMENTAL GROUPS

GLOSSARY

Fundamental group: In a path-connected topological space X, choose an arbitrary base point p. Then the homotopy classes of loops in X based at p, i.e., continuous functions $f : [0,1] \to X$ with $f(0) = f(1) = p$ may be endowed with the structure of a group, where the group operation is a concatenation of two loops at double speed. This is called the fundamental group $\pi_1(X)$, and up to isomorphism it does not depend on the choice of base point p. If $\pi_1(X) = 0$ then X is said to be simply connected. The first homology group $H_1(X, \mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X)$.

A chain of implications: The following implications hold for an arbitrary simplicial complex X.

$$\pi_1(X) = 0 \implies H_1(X, \mathbb{Z}) = 0 \implies H_1(X, \mathbb{Z}/q\mathbb{Z}) = 0 \implies H_1(X, \mathbb{R}) = 0.$$

Here q is any prime. This is a standard application of the universal coefficient theorem for homology [Hat02].

A partial converse to one of the implications is the following. If $H_1(X, \mathbb{Z}/q\mathbb{Z}) = 0$ for every prime q, then $H_1(X, \mathbb{Z}) = 0$.

Hyperbolic group: A finitely presented group is said to be word hyperbolic if it can be equipped with a word metric satisfying certain characteristics of hyperbolic geometry [Gro87].

Kazhdan's property (T):   A group G is said to have property (T) if the trivial representation is an isolated point in the unitary dual equipped with the Fell topology. Equivalently, if a representation has almost invariant vectors then it has invariant vectors.

Group cohomology: Associated with a finitely presented group G is a contractible CW complex EG on which G acts freely. The quotient BG is the classifying space for principle G bundles. The group cohomology of G is equivalent to the cohomology of BG.

Cohomological dimension: The cohomological dimension of a group G, denoted cdim G, is the largest dimension k such that $H^k(G, R) \neq 0$ for some coefficient ring R.

The random fundamental group $\pi_1(Y(n,p))$ may fruitfully be compared to other models of random group studied earlier, such as Gromov's density model [Oll05]. The techniques and flavor of the subject owes as much to geometric group theory as to combinatorics.

The vanishing threshold and hyperbolicity

Babson, Hoffman, and Kahle showed that the vanishing threshold for simple connectivity is much larger than the homology-vanishing threshold.

THEOREM 22.3.20   [BHK11]

Let $\epsilon > 0$ be fixed and $Y = Y(n, p)$. If

$$p \geq \frac{n^\epsilon}{\sqrt{n}}$$

then w.h.p. $\pi_1(Y) = 0$, and if

$$p \leq \frac{n^{-\epsilon}}{\sqrt{n}}$$

then w.h.p. $\pi_1(Y)$ is a nontrivial hyperbolic group.

Most of the work in proving Theorem 22.3.20 is showing that, on the sparse side of the threshold, π_1 is hyperbolic. This in turn depends on a local-to-global principle for hyperbolicity due to Gromov [Gro87].

Gundert and Wagner showed that it suffices to assume that

$$p \geq \frac{C}{\sqrt{n}}$$

for some constant $C > 0$ to show that w.h.p. $\pi_1(Y) = 0$ [GW16]. Korándi, Peled, and Sudakov showed that it suffices to take $C = 1/2$ [KPS16].

The author suspects that there is a sharp threshold for simple connectivity at $C/\sqrt{n}$ for some $C > 0$.

CONJECTURE 22.3.21   *A sharp vanishing threshold for $\pi_1(Y)$.*

There exists some constant $C > 0$ such that if

$$p \geq \frac{C + \epsilon}{\sqrt{n}},$$

with high probability, $\pi_1(Y) = 0$; and if

$$p \leq \frac{C - \epsilon}{\sqrt{n}},$$

with high probability, $\pi_1(Y) \neq 0$.

Kazhdan's property (T)

One of the most important properties studied in geometric group theory is property (T). Loosely speaking, a group is (T) if it does not have many unitary representations. Property (T) is also closely related to the study of expander graphs. For a comprehensive overview of the subject, see the monograph [BHV08].

Inspired by Garland's method, Żuk gave a spectral condition sufficient to imply (T). Hoffman, Kahle, and Paquette applied Żuk's condition, together with Theorem 22.3.7 to show that the threshold for $\pi_1(Y)$ to be (T) coincides with the Linial–Meshulam homology-vanishing threshold.

THEOREM 22.3.22   [HKP12]

Let $Y = Y(n,p)$.

- *If*

$$p \geq \frac{2\log n + \omega(1)}{n}$$

  then w.h.p. $\pi_1(Y)$ is (T), and

- *if*

$$p \leq \frac{2\log n - \omega(1)}{n}$$

  then w.h.p. $\pi_1(Y)$ is not (T).

Cohomological dimension

Costa and Farber [CCFK12] studied the cohomological dimension of the random fundamental group in [CF13]. Their main findings are that there are regimes when the cohomological dimension is 1, 2, and ∞, before the collapse of the group at $p = 1/\sqrt{n}$.

THEOREM 22.3.23   [HKP12]

Let $Y = Y(n,p)$.

- *If*

$$p \ll \frac{1}{n}$$

  then w.h.p. cdim $\pi_1(Y) = 1$ [CCFK12].

- *If*

$$\frac{3}{n} \leq p \ll n^{-3/5}$$

  then w.h.p. cdim $\pi_1(Y) = 2$ [CF15a] .

- *if*

$$n^{-3/5} \ll p \leq n^{-1/2-\epsilon}$$

  then w.h.p. cdim $\pi_1(Y) = \infty$ [CF13].

Here we use $f \ll g$ to mean $f = o(g)$, i.e.,

$$\lim_{n\to\infty} f/g = 0.$$

Newman recently refined part of this picture [New16], showing that if $p < 2.455/n$ w.h.p. cdim $\pi_1(Y) = 1$, and if $p > 2.754/n$ then w.h.p. cdim $\pi_1(Y) = 2$. The precise constants are c_2 and c_2^*, defined in Section 22.3.2.

The fundamental group of the clique complex

Babson showed that $p = n^{-1/3}$ is the vanishing threshold for $\pi_1(X(n,p))$ in [Bab12]. An independent and self-contained proof, including more refined results regarding torsion and cohomological dimension, was given by Costa, Farber, and Horak in [CFH15].

Finite quotients

Meshulam studied finite quotients of the random fundamental group, and showed that if they exist then the index must be large—the index must tend to infinity with n. His technique is a version of the cocycle-counting arguments in [LM06] and [MW09], for non-abelian cohomology.

THEOREM 22.3.24   Meshulam, [Mes13]

Let $c > 0$ be fixed. If $p \geq \frac{(6+7c)\log n}{n}$ then w.h.p. $\pi_1(Y)$ has no finite quotients with index less than n^c. Moreover, if H is any fixed *finite group and $p \geq \frac{(2+c)\log n}{n}$ then w.h.p. there are no nontrivial maps to H.*

22.3.5 PHASE TRANSITIONS IN THE MULTI-PARAMETER MODEL

Applying Garland's method, Fowler described the homology-vanishing phase transition in the multi-parameter model in [Fow15].

THEOREM 22.3.25   Fowler [Fow15]

Let $X = X(n, p_1, p_2, \dots)$ with $p_i = n^{-\alpha_i}$ and $\alpha_i \geq 0$ for all i. If

$$\sum_{i=1}^{k} \alpha_i \binom{k}{i} < 1,$$

then w.h.p. $H^{k-1}(X, \mathbb{Q}) = 0$. If

$$\sum_{i=1}^{k} \alpha_i \binom{k}{i} \geq 1$$

and

$$\sum_{i=1}^{k-1} \alpha_i \binom{k-1}{i} < 1$$

then w.h.p. $H^{k-1}(X, \mathbb{Q}) \neq 0$.

22.4 BETTI NUMBERS AND PERSISTENT HOMOLOGY

22.4.1 BETTI NUMBERS

The random clique complex

In the random clique complex $X(n, p)$, it was noted in [Kah09] that if

$$1/n^{1/k} \ll p \ll 1/n^{1/(k+1)},$$

then

$$\mathbb{E}\left[\beta_k\right] = (1 - o(1)) \binom{n}{k+1} p^{\binom{k+1}{2}}.$$
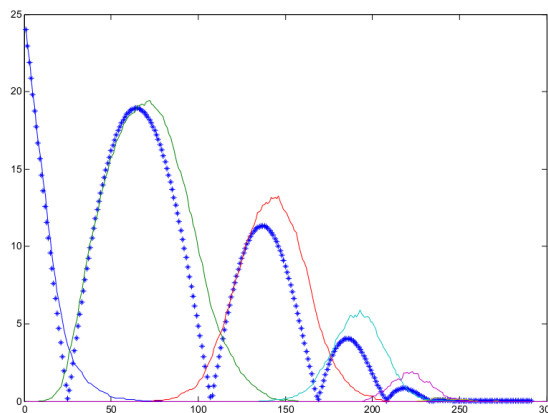
FIGURE 22.4.1
$|\mathbb{E}[\chi]|$ *plotted in blue, against the Betti numbers of the random flag complex $X(n,p)$. Here $n = 25$ and p varies from 0 to 1. The horizontal axis is the number of edges. Computation and image courtesy of Vidit Nanda.*

A more refined estimate may be obtained along the following lines.

We have the Euler relation

$$\chi = f_0 - f_1 + f_2 - \cdots = \beta_0 - \beta_1 + \beta_2 - \ldots,$$

where f_i denotes the number of i-dimensional faces. The expected number of i-dimensional faces is easy to compute—by linearity of expectation we have

$$\mathbb{E}[f_i] = \binom{n}{i+1} p^{\binom{i+1}{2}}.$$

If we make the simplifying assumption that only one Betti number β_i is nonzero, then we have

$$|\chi| = |f_0 - f_1 + f_2 - \ldots| = \beta_i.$$

So we obtain a plot of all of the Betti numbers by plotting the single function

$$|\chi| = |f_0 - f_1 + f_2 - \ldots|$$
$$= \left| \binom{n}{1} - \binom{n}{2} p^1 + \binom{n}{3} p^2 - \binom{n}{4} p^6 + \ldots \right|.$$

This seems to work well in practice. See for example Figure 22.4.1. It is interesting that even though all of the theorems we have discussed are asymptotic as $n \to \infty$, the above heuristic gives a reasonable prediction of the shape of the Betti number curves, even for $n = 25$ and $p \le 0.6$.

Homological domination in the multi-parameter model

In [CF15b, CF17], Costa and Farber show that for many choices of parameter (an open, dense subset of the set of allowable vectors of exponents) in the multi-parameter model, the homology is dominated in one degree.

Random geometric complexes

Betti numbers of random geometric complexes were first studied by Robins in [Rob06].

Betti numbers of random geometric complexes are also studied in [Kah11]. Estimates are obtained for the Betti numbers in the subcritical regime $nr^d \to 0$. In this regime the Vietoris–Rips complex and Čech complex have small connected components (bounded in size), so all the topology is local.

For the following theorems, we assume that n points are chosen i.i.d. according to a probability measure on $\mathbb{R}^d$ with a bounded measurable density function f. So the assumptions on the underlying probability distribution are fairly mild.

The following describes the expectation of the Betti numbers of the Vietoris–Rips complex in the subcritical regime. In this regime the homology of $VR(n, r)$ is dominated by subcomplexes combinatorially isomorphic to the boundary of the cross-polytope.

THEOREM 22.4.1 [Kah11]

Fix $d \geq 2$ and $k \geq 1$. If $nr^d \to 0$ then

$$\mathbb{E}\left[\beta_k[VR(n, r)]\right] / \left(n^{2k+2} r^{d(2k+1)}\right) \to C_k,$$

as $n \to \infty$ where D_k is a constant which depends on k, d, and the function f.

The analogous story for the Čech complex is the following. Here the homology is dominated by simplex boundaries.

THEOREM 22.4.2 [Kah11]

Fix $d \geq 2$ and $1 \leq k \leq d - 1$. If $nr^d \to 0$ then

$$\mathbb{E}\left[\beta_k[C(n, r)]\right] / \left(n^{k+2} r^{d(k+1)}\right) \to C_k,$$

as $n \to \infty$ where D_k is a constant which depends on k, d, and the function f.

In the thermodynamic limit

The thermodynamic limit, or critical regime, is when $nr^d \to C$ for some constant $C > 0$. In [Kah11], it is shown that for every $1 \leq k \leq d - 1$, we have $\beta_k = \Theta(n)$. Yogeshwaran, Subag, and Adler obtained the strongest results so far for Betti numbers in the thermodynamic limit, including strong laws of large numbers [YSA15], in particular that $\beta_k/n \to C_k$.

Limit theorems

Kahle and Meckes computed variance of the Betti numbers, and proved Poisson and normal limiting distributions for Betti numbers in the subcritical regime $r = o\left(n^{-1/d}\right)$ in [KM13].

More general point processes

Yogeshwaran and Adler obtained similar results for Betti numbers in a much more general setting of stationary point processes [YA15].

22.4.2 PERSISTENT HOMOLOGY

Bubenik and Kim studied persistent homology for i.i.d. random points on the circle, in the context of a larger discussion about foundations for topological statistics [BK07].

Bobrowski, Kahle, and Skraba studied maximally persistent cycles in $VR(n,r)$ and $C(n,r)$. They defined the *persistence* of a cycle as $p(\sigma) = d(\sigma)/b(\sigma)$, and found a law of the iterated logarithm for maximal persistence.

THEOREM 22.4.3   [BKS15]

Fix $d \geq 2$, and $1 \leq i \leq d - 1$. Choose n points i.i.d. uniformly randomly in the cube $[0,1]^d$. With high probability, the maximally persistent cycle has persistence

$$\max_{\sigma} p(\sigma) = \Theta \left(\frac{\log n}{\log \log n} \right)^{1/i}.$$

CONJECTURE 22.4.4   *A law of large numbers for persistent homology.*

$$\max_{\sigma} p(\sigma) / \left(\frac{\log n}{\log \log n} \right)^{1/i} \to C,$$

for some constant $C = C_{d,i}$.

OTHER RESOURCES

For an earlier survey of Erdős–Rényi based models with a focus on the cohomology-vanishing phase transition, see also [Kah14b]. For a more comprehensive overview of random geometric complexes, see [BK14].

Several other models of random topological space have been studied. Ollivier's survey [Oll05] provides a comprehensive introduction to random groups, especially to Gromov's density random groups and the triangular model. Dunfield and Thurston introduced a new model of random 3-manifold [DT06] which has been well studied since then.

RELATED CHAPTERS

Chapter 21: Topological methods in discrete geometry
Chapter 24: Persistent homology
Chapter 25: High-dimensional topological data analysis

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