1. From class notes!

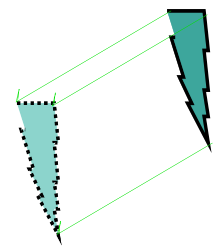
A **transformation** is a one-to-one mapping of the points of the plane to new points of the same plane.

An **isometry**, also called a "**rigid motion**", is a transformation which preserves distances.
Preserving all distances preserves figures (think of triangles).

There are *only* four types of isometries of the plane:

Translation	("Slide")
Reflection	("Flip")
Rotation	("Turn")
Glide reflection	("Flip'nSlide")

Translation

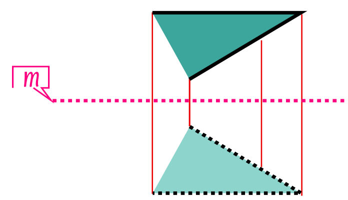


•Determined by a vector (an arrow with specific length and direction)

•Moves all points of the plane in one direction, the same distance... determined by the "slide arrow" or vector of the translation.

•Since all points move the same direction, points move on parallel paths.

Reflection

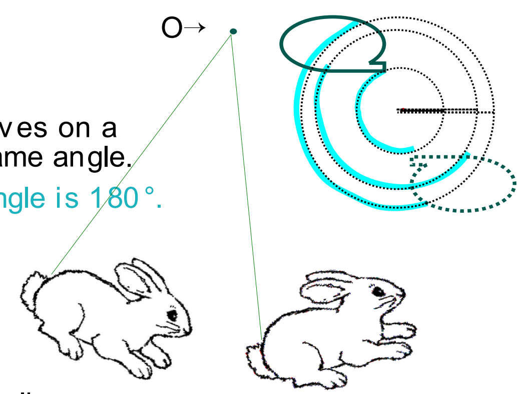


•Except for those on the line of reflection, all points move across the line of reflection (perpendicular to the line of reflection); points equally distant from the line of reflection, but on opposite sides, essentially swap places.

•The reflection line is the ⊥ bisector of the segment joining a point and its image.

•Clockwise vs counter-clockwise sense/orientation reverses (ie figures "flip").

Rotation



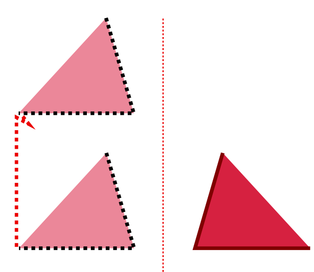
•Determined by a center and directed angle of rotation

•Every point in the plane, except the center of rotation, moves on a circular path around the center of rotation, through the same angle.

•The center of rotation stays fixed. In example at right, angle is 180°.

The angle of rotation for the bunnies is about 42°.

Glide-Reflection



•Determined by a line of reflection and vector parallel to the line.

•All points of the plane flip across the line of reflection, then "glide".

•No point stays fixed.

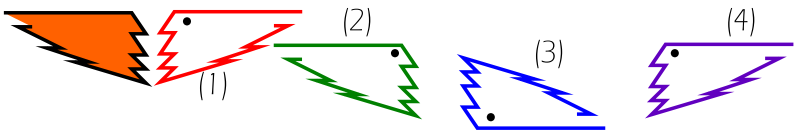
•The reflection line *contains the midpoints* between points and their images.

•Clockwise vs counter-clockwise sense (orientation) reverses. (i.e. figures "flip".)

What type of isometry is it?
✅Find at least three pairs of matching points, and name them, e.g. ABC and A'B'C'.
✅Check the orientation of the figure & image. If path ABC is clockwise & A'B'C' is counter-clockwise, then orientation reversed, and the isometry must be a Reflection or Glide-Reflection. If the orientation is not reversed, then the isometry is a translation or rotation.
Draw arrows from A to A', B to B'.
If they are the same (length & direction), the isometry is a translation.
If they are one direction, but different lengths, the isometry must be a reflection.
If they differ in direction, the isometry is either a rotation or glide-reflection.

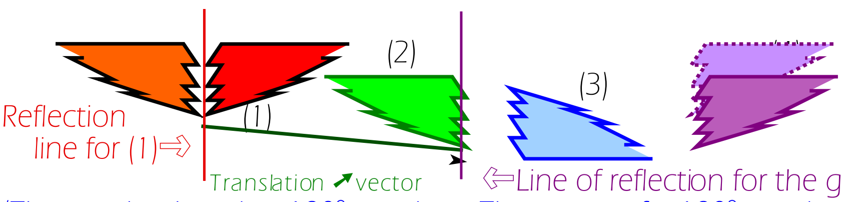
2. See the exercises using multiple reflections on the TG-2 and TG-3 worksheets.

3. Translations leave a figure "facing the same direction". All other transformations can change the direction a figure is facing (although this may not be obvious in special circumstances, such as a 180° rotation of a symmetric figure, e.g. a rectangle). A figure that is "facing NW" will remain "facing NW" after any number of translations of the plane. The figure never gets to "turn", as it does in a rotation. As for reflections, we have seen how two reflections can cause a rotation, and just one reflection results in the figure "facing the opposite direction".



Q: Which of the four images is facing the same direction as the original (shaded) ?
A: only the second one.
Q: What are these transformations; Can you identify them completely?

A. The transformations are (1) Reflection (2) Translation (3) Rotation (4) Glide-reflection



After reflection, before the glide, the ⇦ image is here.

(The rotation here is a 180° rotation. The center of a 180° rotation is always easy to find, because it is halfway between a point and its image. (Find a point on the original figure, and its image on the new figure. The midpoint is it. This works onl y for 180° rotations.) Other rotations require more effort to locate the center.

4. Only Reflections and Glide-reflections change the clockwise sense of a figure. That is because only these types of transformations "flip" the plane over, and "flipping" is required to reverse the clockwise sense of the figure. The effects on a figure, of translating and rotating the plane can be visualized by sliding and turning a figure on a flat surface, and neither of these results in a "flipped over" figure.

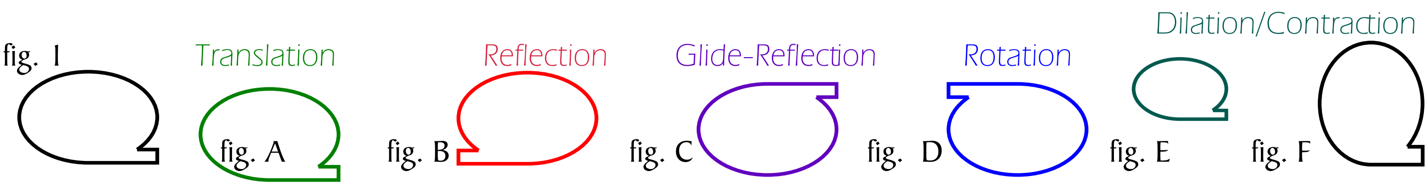
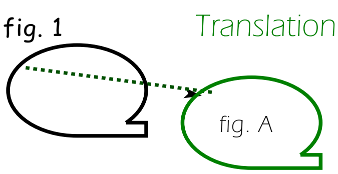
5. 

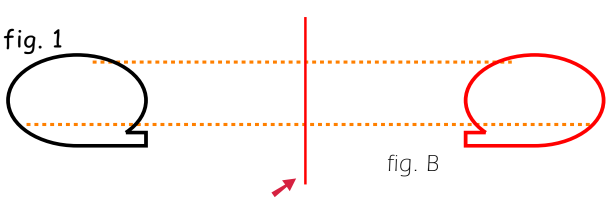
Figure E is the result of a dilation, a "size transformation", not an isometry (not rigid motion).
Figure F is not even similar to Figure 1. We do not study any such transformations in this course.

More details:

5. 

Showing the vector of translation.

Notice that figure A faces the same direction as the original figure (fig. 1). Also notice all points move the same distance and direction.

5. 

Showing the line of reflection.

Notice that all points move on parallel paths, but different distances....

5. fig. 1

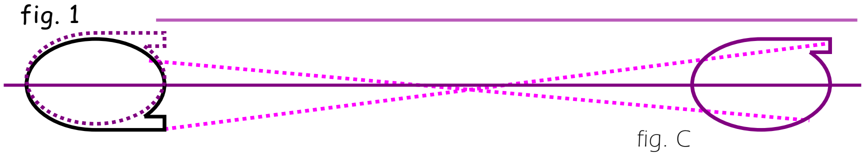
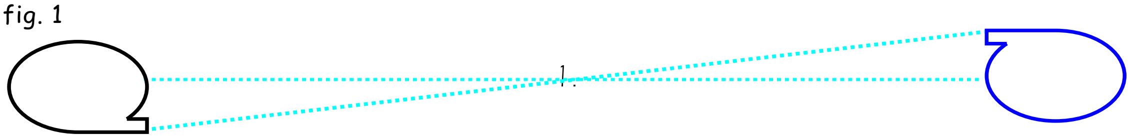


fig. C

Showing the vector of translation, and the line of reflection.

Notice points move on non-parallel paths. ... and orientation is reversed (clockwise-> counterclockwise).

5. fig. 1



Finding the center of rotation is trivial in the case of 180° rotation.

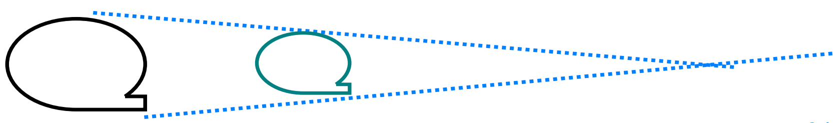
The angle of rotation is 180°

fig. D

Notice that points move on non-parallel paths,
but orientation is unchanged... what was clockwise, remains clockwise.

fig. E

5. fig. 1

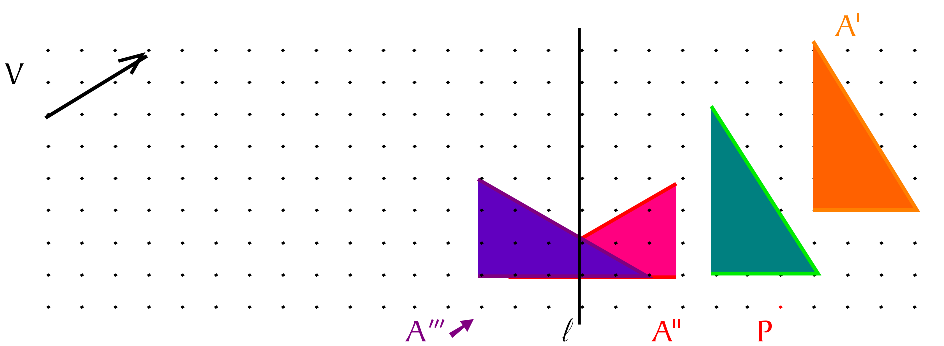


The scale factor is about ⅔.

Figure E can be obtained from fig. 1 by a contraction of the plane (dilation in reverse), then translating.
Don't stress over the "contraction" terminology.

By the way, when we place figures 1 & E in a "perspective" arrangement, fig 1 is about 3/2 as far away from the center as fig E. So the dilation factor from fig. E to fig. 1 is 3/2 (or 1.5 if you prefer decimal form). Thus the contraction factor from fig. 1 to fig. E is the opposite ratio, 2/3.

6.



Translate the figure A (green) by vector V, label that A'.

Then rotate A' 90° about P ; label that A''.

Then reflect A'' through line ℓ . Label the final result A'''.

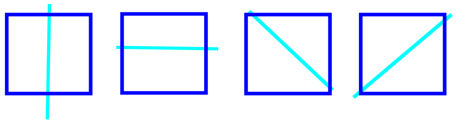
Hints for getting this right:

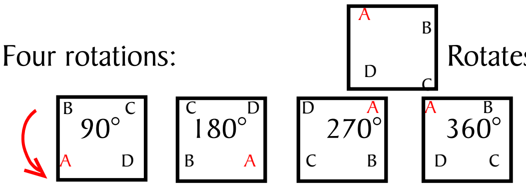
For the translation, locate the new triangle vertices by counting squares. Since V points 3 "east", 2 "north", each vertiex should move likewise... 3 squares right, 1 up.

For the rotation:
90° turns are easy to pinpoint on the grid.
Consider the lower left vertex of the triangle A′, which lies 1 square right and 3 squares above P.
When the plane rotates 90° around P, the new vertex will lie 3 squares above and 1 square left of P.

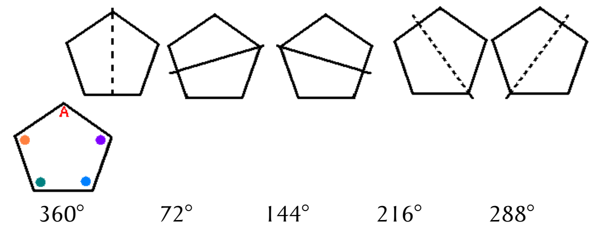
For the reflection:
Consider the top right vertex of triangle A″, which lies 3 squares right of reflection line ℓ .
The new vertex is exactly opposite across the line ℓ , and so lies 3 squares left of reflection line ℓ .

7. 7a. Symmetries of a square– Four line symmetries and Four rotations: Rotates to...

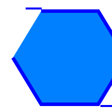




7b. There are five line symmetries of the regular pentagon, one from each vertex through the midpoint of the opposite side.
There are also five rotational symmetries of the regular pentagon; the smallest is one-fifth of a rotation, 72°;
the other four are, of course, 144°, 216°, 288° and 360°.



7c. The regular hexagon has six <u>line</u>, and six <u>rotational</u>, symmetries.
However, the figure given has no line symmetries and only 180° and 360° rotational symmetries, because of the "flags" that have been added.



7d. About the individual letters in "S Y M M E T R I C" :

The letter S, as printed, is like the letters N and Z– although at first glance S may appear to have some line symmetry, it has none. What we are noticing is the 180° rotational symmetry that all these letters (N, S, & Z) have. Of course there is also the 360° rotation.

The letter I has two line symmetries (it can be flipped on the horizontal and vertical axes and land back on top of its original footprint). This letter also has 180° and 360° rotational symmetries.

The letter R has no line nor rotational symmetry.

Y and M and T each have one line symmetry, about their vertical axes. No rotational symmetry.

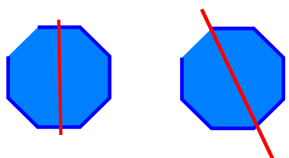
E and C both have one line symmetry, about their horizontal axes. No rotational symmetry.

7e.

No line symmetry

180° rotational symmetry
(& 360° of course)

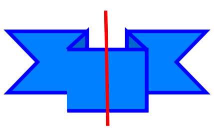
f.



Type "A" Type "B"

The regular octagon has eight line symmetries–
Four of type A, from midpoint to opposite midpoint.
Four of type B from vertex to opposite vertex.
(As is the case with the square... one may generalize)
In addition the regular octagon has eight rotational symmetries, rotating (about the center of course) by 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°

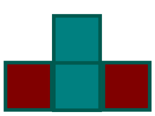
7g.



Ribbon has one line symmetry,
About its vertical axis. .
No rotational symmetry

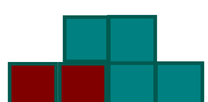
8. There are numerous ways to add squares (congruent to existing squares) to each of the following figures to make the figure have a line symmetry, but no rotational symmetry.

a.



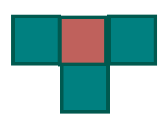
The given figure had line & rotational symmetry. Squares added to destroy the rotational symmetry.

b.



The given figure had rotational symmetry, but not line symmetry. Two added squares destroy the Rotational sym. while adding line.

c.



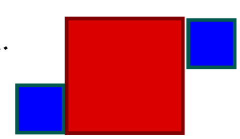
The given figure already had the required properties. The optional added square preserves them.

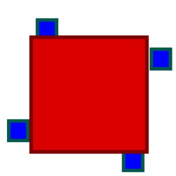
9. Add to the figure at right so the resulting figure has rotational symmetry but no line symmetry.

Again, there are numerous ways to do this. Two examples are shown for each figure.

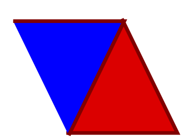
These second alternatives assume the figures were regular polygons.

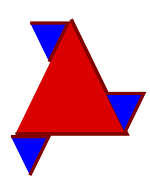
a.



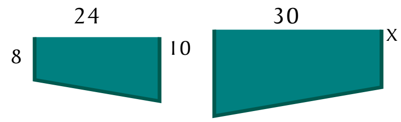


b.





10. Given the figures at right are similar,
a. What is the scale factor from the left figure to the right?
b. Find x .

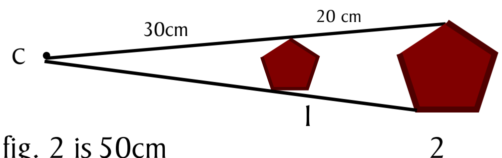


a. Scale factor is 30/24 = 5/4 or 1.25

b. x/8 = 30/24
x = 8• 30/24 = 10

Or you could just say x= 8•1.25

11. In a dilation of the plane with center C, the distances between C and a figure and its image are as shown. What is the scale factor of the transformation?

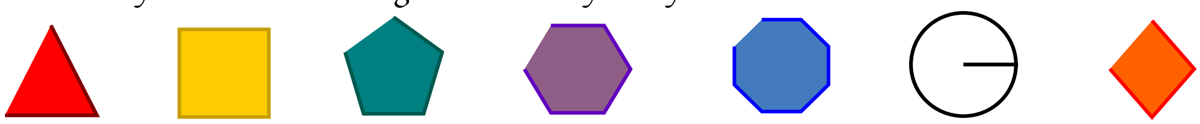


The distance between C and the top of fig. 2 is 50cm
The distance between C and the top of fig. 1 is 30cm

The ratio of any distance in fig. 2 to the corresponding distance in fig. 1 ...must be the same as the ratio of those line segments: 5/3. This constant ratio is the "scale factor" relating size of fig.2 to fig.1 .

A1. How many rotational symmetries has each figure? How many line symmetries?

Assuming the first five figures are regular



Rotational:	3	4	5	6	8	∞	2
(smallest angle):	120°	90°	72°	60°	see #7f	No smallest!	180°
Lines :	3	4	5	6	8	∞	2

A2. Yes. Yes. Yes. These questions merely restate question #2, but this time with hints as to the detail. Answers are demonstrated by exercises C1 and M5 and C5 on pages TG-2 and TG-4.

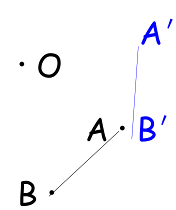
A3. These questions are the logical extension of question #2, and the groundwork for justification of the answers is laid in question #3.

Translations cannot be used to generate all rigid transformations, because no figure is ever turned (so no rotations) and no figure is ever flipped (no clock-sense reversals) (so no reflections nor glide-reflections).

Rotations likewise cannot do the job, because, like translations, rotations never flip figures– never reverse the clock-sense, thus it is impossible to generate any reflections or glide-reflections by any series of rotations.

Reflections were previously shown to do the job; glide-reflections can too.

A4. The rotated segment will appear approximately as A'B'.
Use tracing paper to check that rotating around O really moves AB to A'B'.
Draw angle BOB' or AOA' and use a protractor to check the measure of the angle.



A5. a. T b. T c. Rotation d. GR

A6. a. Always a T (Never a R or M or GR) b. T perpendicular to the mirror lines, and twice the distance between.
c. R about the same center d. T R M GR (all!!) e. T R (but never M or GR)

A7. Generally the order makes a difference (unless the centers are the same).

A8. a–d. T e–g. R h–i. M j–k. GR