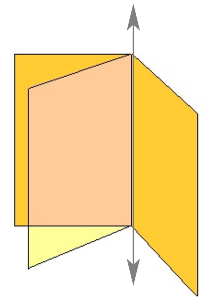
Math 310 ★ Fall 2006 ★ Test #2 ★ 100 points ★ ANSWERS !

1a. The number of distinct points necessary to determine a specific line is 2

"Two distinct points determine a line." Ie "through any two points there is one and only one line."

How many lines can YOU draw through the two points at right?
How many lines can be drawn through one single point?

b. The number of planes containing a line is infinite.



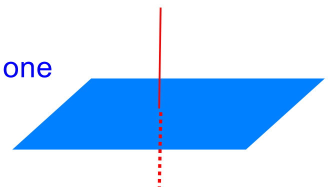
Consider the line at the corner of the room, where two walls meet . . . .
Each wall represents a different plane through the line.
A wall could be built out from that corner in any direction.

c. The number of intersection points of a pair of skew lines is 0 (none).

Skew lines never meet; they exist in different planes.
An example of skew lines: a line where the west wall of the room meets the ceiling, and the line where the north wall meets the floor.

d. The number of points shared by a plane and a line perpendicular to the plane is one

(where the line pierces the plane, as a needle might pierce a piece of paper)

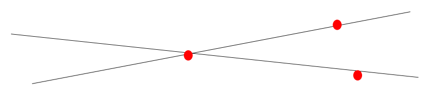


e. The number of non-collinear points needed to determine a plane is 3

Three non-collinear points determine a plane.

That's why a three-legged stool can sit solidly on a floor, while a two-legged stool would fall over... and why a four-legged chair sometimes rocks (when the fourth leg does not end in the plane determined by the first three leg bottoms). That's why a camera is often mounted on a TRIpod!

f. The number of planes containing two lines which intersect is one.



The point where the lines intersect, together with one additional point from each line, comprise three noncollinear points, enough to determine a plane. Then, because two points of each line belong to the plane, each entire line must lie in the plane.

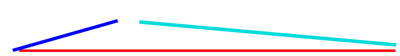
g. The number of different (non-congruent) triangles with sides 5cm, 7cm, & 9cm, is 1.

There is only one size & shape triangle having sides 5cm, 7cm, & 9cm. By the "SSS" theorem of triangle congruence, there can be only one (all others are congruent). And since $9 \leq 5+7$, these values satisfy the triangle inequality, and so there is such a triangle.

h. The number of different (non-congruent) triangles with sides 2cm, 4cm, & 6cm, is 0 (none) .

You might say to yourself, this was a trick question. But in fact, having this question right after part g above was intended to make you stop and think (why are there two of these questions? There must be something different going on here). What is different here is the failure to meet the requirements of the triangle inequality. To have a genuine triangle, surrounding some interior points, the two shortest segments must total more than the longest segment.

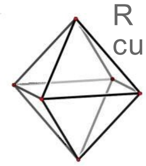
When we attempt to build a triangle using these lengths, we see it never gets off the ground! The 2cm & 4cm segments are just enough to span 6cm when laid flat. We need more length to "tent up" these segments and have them meet!



2. Of the choices given the BEST completion of each statement follows:

A circle	B cube	C hexahedron	D line	E octahedron	F parallelogram
G plane	H point	I polygon	J polyhedron	K prism	L pyramid
M rectangle	N rhombus	O segment	P sphere	Q square	R simple closed curve

a. A quadrilateral with all sides congruent is a RHOMBUS.

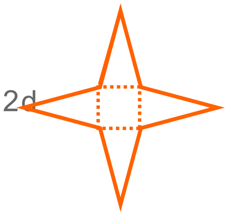


2b

b. The polyhedron illustrated at right is a (regular) OCTAHEDRON.

c. The set of all points in a plane equally distant from a given point *P* is a CIRCLE.

2d

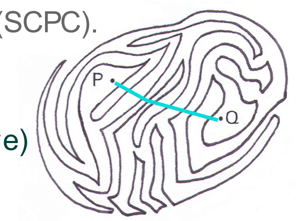


d. The figure at right can be folded up into the polyhedron known as a PYRAMID.

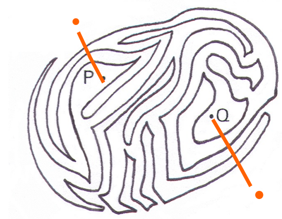
e. A simple closed curve consisting of line segments is a POLYGON.

f. A parallelogram with an interior angle measuring 90° is a RECTANGLE.

3. In the illustration at right, P & Q are tangled with a simple closed plane curve (SCPC). Are the points P and Q on the same side of the curve?

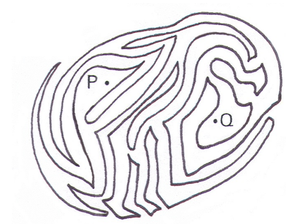


We can tell that P & Q are on opposite sides of the curve in several ways. Probably the most direct is to connect them with a segment (OR a simple curve) and see that the segment crosses the given SCPC an odd number of times, showing, by the Jordan Curve Theorem, that P & Q are on opposite sides– one is interior and the other exterior to the curve.

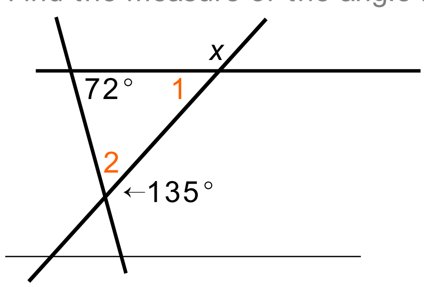


Another, similar method: Connect P to an exterior point. The connecting curve crosses the given SCPC an odd number of times, showing that P is in the INTERIOR of the curve. Similarly, Q is connected to an exterior point, and the even number of SCPC crossings shows us Q is EXTERIOR.

Another method is to color in the interior of the curve (pick an interior point to start, and obey the lines)...
Or color the exterior inward.
Try it on the clean copy at right.



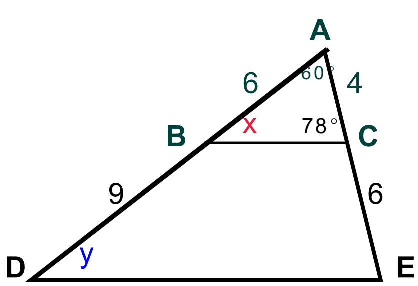
4. Find the measure of the angle marked *x*.



Angle 2 is the supplement of the 135° angle, which makes $m(\angle 2) = 180° - 135° = 45°$.
$m(\angle x) =$ sum of the two remote interior angles, $72° + 45° = 117°$

But most people would probably find $m(\angle 1)$:
$m(\angle 1) = 180° - (72° + 45°) = 180° - 117° = 63°$
then find $m(\angle x) = 180° - m((\angle 1) = 180° - 63° = 117°$

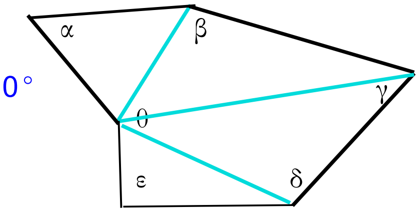
5. Given the triangle and measurements illustrated, **find** the measure of the angle marked *y*. **Explain** what triangles are similar and how you know they are.



△ADE ~ △ABC because the sides of ADE are cut into segments of the same proportions (6:9 as 4:6 —both are in the ratio 2:3) which tells us AC is parallel to DE, thus forming the same angles. (This can also be argued on the basis of SAS: AC:AD is 6:15 and AC:AE is 4:10 – both reduce to the ratio 2:5 between those sides; and angle A is shared, therefore congruent.)

Knowing those two triangles are similar tells us angle D (marked y) is congruent to angle x, which must be $180° - (60° + 78°) = 180° - 138° = 42°$

6. *Without* using a protractor, *showing your work*, find the **sum** of the measures of the interior angles in the polygon at right:



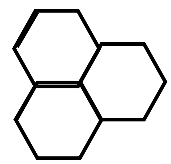
$m(\angle\alpha) + m(\angle\beta) + m(\angle\gamma) + m(\angle\delta) + m(\angle\varepsilon) + m(\angle\theta) = 4 * 180° = 720°$

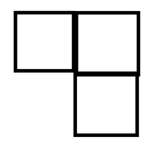
We triangulate the region inside the polygon.... and see four triangles covering the region, while sharing the vertices (& thus the interior angles) of the original polygon.

What is the measure of one interior angle of a **regular hexagon**?

The total of the measures of the interior angles of every hexagon is 720°. In the regular hexagon, each interior angle has the same measure. Thus each must measure $720°/6 = 120°$

7. **Explain** why there can be no regular convex polyhedra with faces which are hexagons. Be specific, but concise.

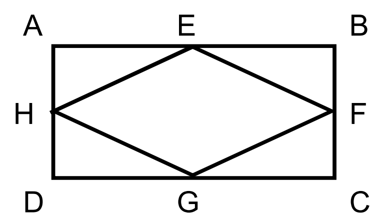




A minimum of three faces must meet at any vertex of a polyhedron. When three regular hexagons meet at a vertex, their interior angles contain 120° angles, which add up to 360°, demonstrating that the hexagons surround the point while lying flat, in a plane. Thus the hexagons are suitable for tiling a floor, but not for creating a 3-D shape. (The sum of the angles must be less than 360°, such as the three squares which meet at the vertex of a cube, whose angles at the vertex sum to 3•90° = 270°. Closing the gap creates the convex corner, or vertex.)

If additional (> 3) hexagons are added, the sum of the angles exceed 360°, so a "ruffle" would be created, and thus the figure could not be convex.

() 8. Given that EFG & H are the midpoints of the sides of rectangle ABCD, prove that quadrilateral EFGH is a rhombus. You can mark the illustration to supplement your statements.



We will show EFGH is a rhombus by showing that the four triangles— AEH, BEF, DGH, and CGF are congruent, so their hypotenuses are congruent.

AB ≅ DC and AD ≅ BC because ABCD is a rectangle, & opposite sides of any rectangle are congruent.

Since E is the midpoint of AB, AE ≅ BE, and the length of each is half the length of AB.

Similarly, DG ≅ CG, and each is half the length of CD... And thus congruent to AE and BE.

(To recap: DG ≅ CG ≅ AE ≅ BE.)

By similar argument, we know AH ≅ DH ≅ BF ≅ CF.

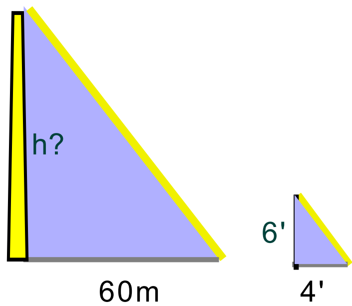
$\angle A \cong \angle B \cong \angle C \cong \angle D$ (each is 90°, since ABCD is a rectangle)

Thus, by the SAS theorem of triangle congruence, $\triangle AEH \cong \triangle BEF \cong \triangle DGH \cong \triangle CGF$.

Corresponding parts of congruent triangles are congruent. Thus the hypotenuses of those four triangles are congruent. This demonstrates that EFGH is a rhombus, by definition*. QED

* (A rhombus is defined to be a quadrilateral with all four sides congruent.)

9. A monument casts a 60-meter shadow at the same time that a 6-foot post casts a shadow 4 feet long. How tall is the monument?



$$\frac{h}{60m} = \frac{6'}{4'}$$ (Notice the 'feet' units reduce, right along with the factor 2....)

$$h = \frac{3}{2}\,60m$$

$$= 90m$$

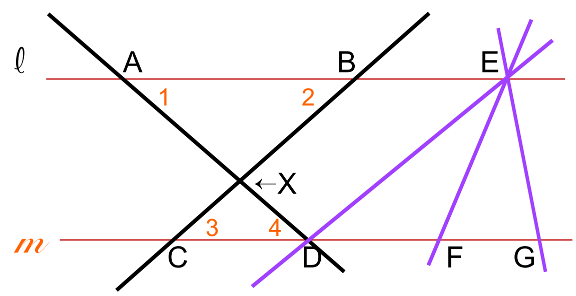
Comment:

Sunlight streams over the monument, and over the post, at the same angle (parallel rays of light). The monument and post are both upright (perpendicular to the ground.) Thus we have congruence of two pairs of angles in the triangles created by horizontal ground, the uprights, and sunlight streaming over the the top. So the triangles are similar, and thus their sides are in the same proportion.

The above statement says: h : 60m = 6' : 4'

But we could just as easily say: h : 6' = 60m : 4'

10. Given $\ell \parallel m$ Name two similar triangles; **explain** concisely what guarantees that the triangles are *similar*.



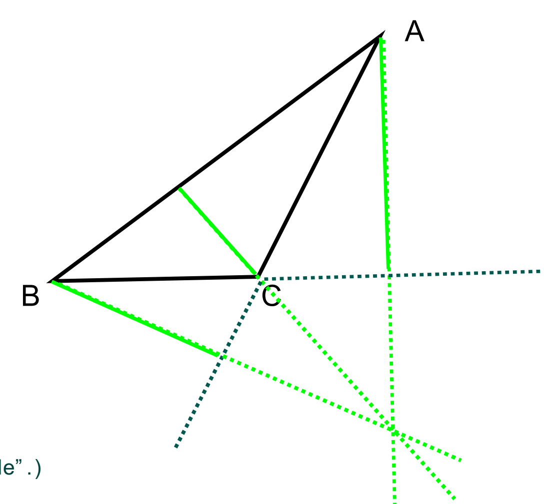
There's a lot of 'stuff' in that sketch.
Let's note that $\ell \parallel m$ and, although DE appears nearly parallel to CB, there is absolutely no justification for making any conclusion about ΔADE in conjunction with any other triangle in this sketch.

Further, also notice that DE and FE and GE have nothing in common other than being concurrent at E; they form three different angles with line m , and there are no similar triangles cleverly hidden there.

The similar triangles are AXB and DXC. Within those triangles, the angles marked 1 and 4 (at A & D !) are congruent, as they are alternate interior angles for two parallel lines (m and ℓ) cut by a transversal (AD). Likewise, the angles marked 2 and 3 (at B & C !) are congruent for the same reason (except the transversal in this case is BC). Thus, by the "AA" theorem of triangle similarity, ΔAXB ~ ΔDXC.

One last note: Since the angle at B (in ΔAXB) matches the angle at C (in ΔDXC), when we name the similar triangles, B and C must occupy the same relative position in the naming. For instance:
It is NOT considered true that ΔAXB ~ ΔCXD.
^ ^

11. Carefully illustrate (sketch) the three altitudes of the triangle ABC. Where do their extensions meet?



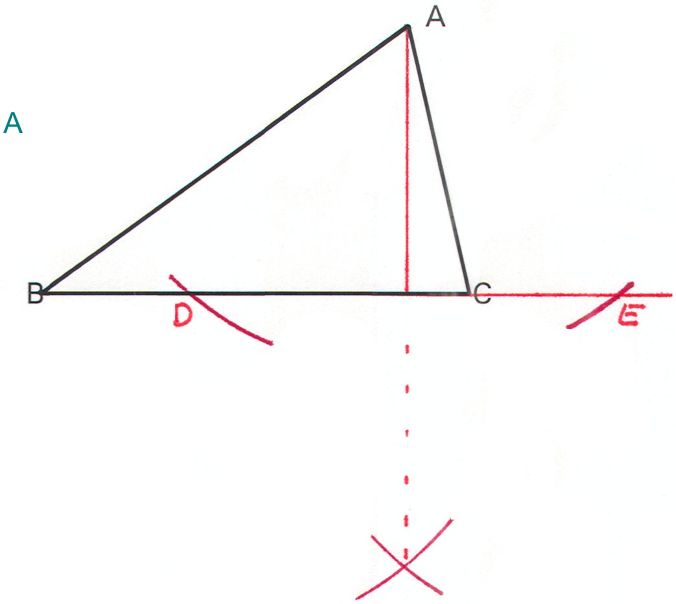
The altitudes (marked ———) extend from each vertex only to the line of the opposite side. Sides AC and BC required extension (············) in order to perceive the foot of the altitudes from B and A, respectively.

The extended altitudes meet at one point, in this case outside the triangle, because it is obtuse.

(You can learn a lot more about this point by looking up "orthocenter of a triangle".)

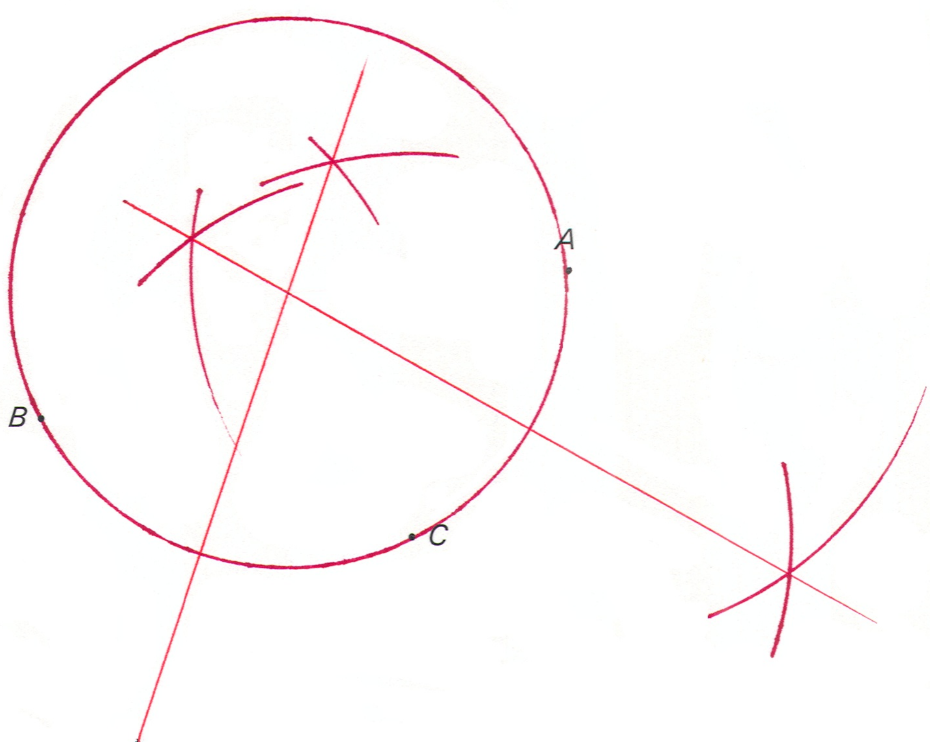
12. *Showing all necessary marks*, **construct the altitude** of ABC that contains the point A.

Drop a perpendicular from A to side BC .

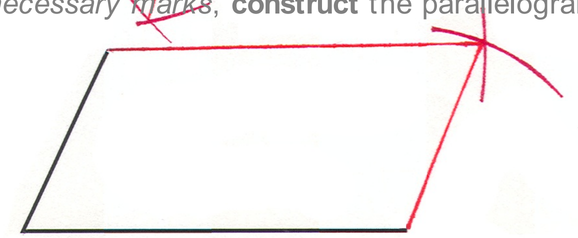


13. *Showing your work,* carefully construct a circle containing the points A, B, and C below.

We construct the perpendicular bisectors of two chords; the chords do not necessarily have to be drawn (the chords used here were AC and BC).



14. *Showing all necessary marks*, **construct** the parallelogram that **contains the sides given**.



Only two arcs are required !
(Because making two new sides congruent to the opposite sides guarantees this is a parallelogram.)