1. A box contains ten marbles: 5 red (R), 3 blue (B), and 2 white (W).
One marble is drawn from the box.
Use the scale below to describe the chance of each of the following events occurring.

NO CHANCE → | Not Likely | Small Chance | Even Chance | Good Chance | Almost Sure | ← CERTAIN

A. A red marble is drawn. EVEN
B. A blue marble is drawn. SMALL chance, or slightly better than ....
C. A white marble is drawn. NOT Likely
D. A black marble is drawn. NO CHANCE !!!
E. A blue or white marble is drawn. EVEN [because three of the six are "winners"]
F. The marble drawn is not white. ALMOST SURE [There's only one way to lose.]
G. The marble drawn is not black. CERTAIN !!!

2. If you were going to turn the above scale into a numeric scale, what values would you assign?

NO CHANCE → | Not Likely | Slim Chance | Even Chance | Good Chance | Almost Sure | ← CERTAIN

3. What numbers would you assign to each of the following events?
A. A red marble is drawn. HALF (half the marbles are winners)
B. A blue marble is drawn. ${}^{2}/_{6}$ or ${}^{1}/_{3}$
C. A white marble is drawn. ${}^{1}/_{6}$
D. A black marble is drawn. $0/6 = 0$ [No Chance → 0 probability]
E. A blue or white marble is drawn. ${}^{3}/_{6}$
F. The marble drawn is not white. ${}^{5}/_{6}$ Also Note this is 1 — P(white) !
G. The marble drawn is not black. ${}^{6}/_{6} = 1$ [Certain → probability = 1.]

4. The possible outcomes in the above experiment could be called: R B W
Some people would argue there are 3 possibilities, so the chance of each must be one-third.
What is wrong with this analysis? How would you try to convince that person otherwise?

These people are ASSUMING that the three outcomes are equally likely to occur.
But R & B & W are NOT equally likely to occur. Assumption is bad!

To clarify the situation, ask doubters to consider the experiment of drawing a marble from a jar containing 100 marbles, with 98 red, 1 blue and 1 white (this just exaggerates the bias toward red). Would they rather bet on getting a RED marble or a WHITE marble?

5. A SAMPLE SPACE for an experiment is the set of all possible outcomes of the experiment.
List two different SAMPLE SPACES (SS) for the above experiment.
A. { R, B, W } B. $\{R_1, R_2, R_3, R_2, R_3, B_1, B_2, B_3, W_1, W_2\}$
Which SS would you rather use to analyze the chance of getting a White marble?
The second SS — it has the advantage that all ten outcomes are EQUALLY LIKELY.

Generally, the most detailed SS is the best— simplest to use,
as it is the one most likely to have equally likely outcomes!

6. LIST SAMPLE SPACES for each of the following experiments:
(use the back of page PN-1)

A. Roll a fair die and see how many dots turn up. SS = { 1, 2, 3, 4, 5, 6 }

B. Toss a fair coin and see what face turns up. SS = { H, T }

C. Toss a fair coin twice and see what turns up each time. { HH, HT, TH, TT }

D. Toss a fair coin three times.... {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

E. Roll a pair of fair dice.... SS =

{ 11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66 }

F. Draw a card from an ordinary deck of cards.
SS is at bottom of this page.

7. In experiment 6A, what is the probability that...

a. the die turns up 4 (dots)? $\frac{1}{6}$

b. the die turns up 7 dots? 0

c. the die turns up an even number of dots? $\frac{3}{6}$

8. In experiment 6F, what is the probability that the card drawn is ...

a. a “face card” (J or Q or K) ? $\frac{12}{52}$ or $\frac{3}{13}$

b. a “heart”? $\frac{13}{52}$ or $\frac{1}{4}$

c. You will win the big prize if the card is a face card or a heart.
What is the probability you will win? $\frac{22}{52}$ or $\frac{11}{26}$

d. What is the probability the card drawn is NOT a Face card or a Heart? $\frac{40}{52}$ or $\frac{10}{13}$

9. In experiment 6E, what is the probability that the sum of the two numbers on the dice ...

a. is 2? $\frac{1}{36}$

b. is 11? $\frac{2}{36}$

c. is 7? $\frac{6}{36}$ or $\frac{1}{6}$

d. is 7 or 11? $\frac{8}{36}$ or $\frac{2}{9}$

e. is even? $\frac{18}{36}$ or $\frac{1}{2}$

(In the game of craps, a sum of 7 or 11 on the first roll is an automatic win.)

10. In experiment 6C, what is the probability that...

a. two Heads turn up? $\frac{1}{4}$

b. one Head and one Tail turn up? $\frac{2}{4}$ or $\frac{1}{2}$

c. the coin does not turn up Heads twice? $\frac{3}{4}$

SS for #6C:

A♥	A♦	A♣	A♠
2♥	2♦	2♣	2♠
3♥	3♦	3♣	3♠
4♥	4♦	4♣	4♠
5♥	5♦	5♣	5♠
6♥	6♦	6♣	6♠
7♥	7♦	7♣	7♠
8♥	8♦	8♣	8♠
9♥	9♦	9♣	9♠
10♥	10♦	10♣	10♠
J ♥	J ♦	J ♣	J ♠
Q♥	Q♦	Q♣	Q♠
K♥	K ♦	K♣	K♠

These are the "ground rules" of probability:

A probability experiment is a repeatable action with a known set of possible outcomes, in which the particular outcome is unknown in advance of each repetition of the experiment.
The set of possible outcomes is the Sample Space.

P1. Probabilities for the outcomes of the entire Sample Space add up to 1.
P2. Probability is never negative.
A zero probability means "can't happen", or "virtually impossible".
P3. Probability never exceeds 1. A probability of 1 means "certain" or "virtually certain".

In addition, we intuitively understand the following aspects of probability:

P4. Probabilities ADD:
Defn: If $A \cap B = \phi$ then A and B are disjoint.
Defn: Events A & B for which $P(A \cap B) = 0$ are mutually exclusive..*

IF A and B are mutually exclusive events for a probability experiment, then

$$P(A \text{ or } B) = P(A) + P(B)$$

(This is one of two so-called "Addition Rules", which follow from the basic rules of probability.)
"The more ways to win, the greater the chance of winning."

P4A. Addition rule for M.E. events: $P(A \cup B) = P(A) + P(B)$.

See how this applies to #9d: P(7 or 11)= $P(7) + P(11) = 6/36 + 2/36$

But some events are not so simply viewed...

A ∩ B means A and B

P4B. The general Addition Rule is: $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

See how this applies to #8c: P(♥ or F) = P(♥) + P(F) — P(♥F) = $13/52 + 12/52 - 3/52 = 22/52$

Addition rule for M.E. events is just a special case of the general Addition rule...
since if A& B are Mutually Exclusive events, $P(A \cap B) = 0$!

P5. Defn: If two events A & B are disjoint, but together make up the entire sample space, then the events are complementary. For instance, in experiment 6A, the events A, that the die turns up 1, and B, that the die turns up more than 1, are complementary. B is the complement of A.

SS = { 1, 2, 3, 4, 5, 6}	P(SS) = $6/6$ = 1
A = { 1 }	P(A) = $1/6$
B = { 2, 3, 4, 5, 6 }	P(B) = $5/6$

In experiment 6E, rolling a pair of fair dice, let's call A the event that the sum of the dice is 12.
We can see that P(A) = $1/36$. What is the probability that the sum of the dice is less than 12?
(Hint: that's the rest of the sample space, the complement of A!) Answer: $35/36$

From the rules of probability observed above, we can say that the probabilities of complementary events must add up to 1. If $\overline{A}$ represents the complement of A, then:

$$P(A) + P(\overline{A}) = 1$$ ...It follows that:

P5. Rule for complementary events: $P(\overline{A}) = 1 - P(A)$

* Since $P(\phi) = 0$, it follows that if A & B are disjoint, then A & B are mutually exclusive. (The reverse is NOT true.)

"The more conditions required for winning, the smaller the chance of winning."

P6. Probabilities MULTIPLY.

Consider the experiment of tossing two coins [e.g. a dime and a nickel].
We have already seen that a two-dimensional table clarifies the sample space .
Having a good working sample space in turn makes it easier to answer questions.

11. What is the chance heads turn up on both coins? $\frac{1}{4}$

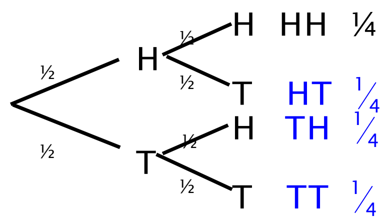
	H	T
H	HH	HT
T	TH	TT

a.

Write a Sample Space. SS = { HH, HT, TH, TT } (SEE THE TABLE! →)
In this SS, are the outcomes equally likely? YES
What is the probability of each outcome? $\frac{1}{4}$
Since each outcome in the SS must be equally likely, and
since there are four outcomes, each must have probability $\frac{1}{4}$.
What is the probability of Heads turning up on both coins? $\frac{1}{4}$

But there is another way to think about such problems, using a tree diagram

b. Use a tree diagram to analyze this experiment.
What is the probability of obtaining heads on both coins? $\frac{1}{4}$



We "get heads half the time" when we toss each coin;
but in two tosses we get two heads
only half of one-half of the time.

b. Complete the tree diagram above right, adding to the list of final outcomes, and showing the probability of each outcome. The tree diagram naturally leads us to the multiplication principle:

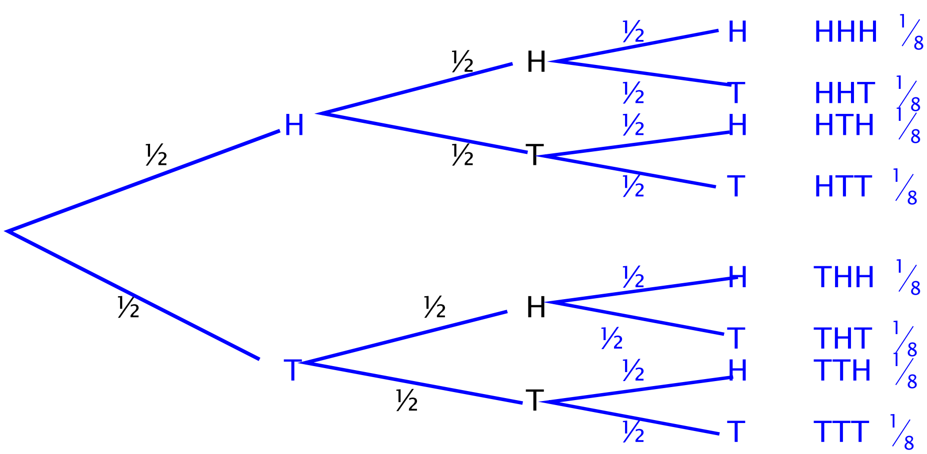
$$P(HH) = P(H\ 1^{st}\ time) \cdot P(H\ 2^{nd}\ time)$$

Probabilities multiply !

In the above example, the probability is relatively simple to find, because the events of obtaining heads on the first and second tosses are independent- neither one affects the other.

> If events A and B are independent, then P(A and B) = P(A) · P(B).
> (Multiplication rule for independent events)

c. Before we explore the meaning of independence, let's draw a tree diagram for the experiment of tossing three coins. Make it complete: include the list of final outcomes and their probabilities.



To win a new car, which game would you choose: toss 1 coin, on which you must get H, or toss 3 coins, on which you must get HHH? Note P(3 heads in a row) = $\frac{1}{8}$, a lot less likely than P(H) = ½ . Conclusion:

"The more conditions required for winning, the smaller the chance of winning."

Now, more about independence.

Given two events from the same sample space, they are either dependent or independent. If neither event affects the other (if one event's occurrence does not change the probability of the other event), then the events are independent. If one has any effect on the other, making it more likely, or less likely, then the events are called dependent... one depends on the other.

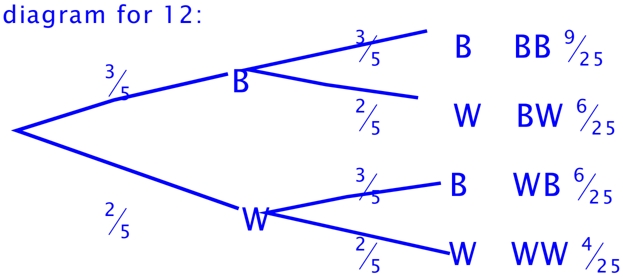
Again: The word "independent" means in terms of probability pretty much what it means in everyday usage: independent events don't depend on, i.e. don't affect, each other. One's occurrence does not make the other's occurrence any more, nor any less, likely (⋆). To better understand independence, we must explore non-independent events, which are called "dependent".

To illustrate, we compare two experiments:
Suppose a jar holds five marbles- 3 blue and 2 white.
Exp. One: we draw two marbles from the jar with replacement.
Exp. Two: we do NOT replace the marbles between draws.   Sample Spaces:

12. In experiment One, what is the probability...
   a. the first marble is blue? $3/5$
   b. the second marble is blue? $9/25 + 6/25 = 3/5$
   c. both marbles are blue? $3/5 \bullet 3/5 = 9/25$
   d. Can the probability in part c be computed the multiplication rule for INDEPENDENT events? Yes $3/5 \bullet 3/5 = 9/25$

	B_1	B_2	B_3	W_1	W_2
B_1	●●	●●	●●	●○	●○
B_2	●●	●●	●●	●○	●○
B_3	●●	●●	●●	●○	●○
W_1	○●	○●	○●	○○	○○
W_2	○●	○●	○●	○○	○○

Tree diagram for 12:

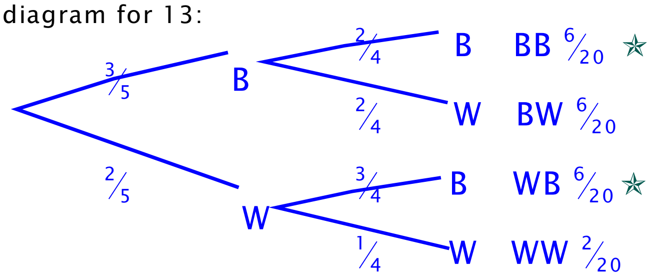


Notice the probabilities in the tree diagram match the proportions of outcomes in the table!

13. In experiment Two, what is the probability
   a. the first marble is blue? $3/5$
   b.* the second marble is blue? $3/5$ (See ✭ below)
   c. both marbles are blue? $3/10$
   d. Can the probability in part c be computed using the multiplication rule for independent events? NO* $3/5 \cdot 3/5 \neq 3/10$

	B_1	B_2	B_3	W_1	W_2
B_1		●●	●●	●○	●○
B_2	●●		●●	●○	●○
B_3	●●	●●		●○	●○
W_1	○●	○●	○●		○○
W_2	○●	○●	○●	○○	

Tree diagram for 13:



Notice again, the probabilities in the tree diagram match the proportions of outcomes in the table!

14. Tree diagrams can help you find the outcomes in the sample space, as well as help find the theoretical probabilities for experiments involving sequential actions.
   a. Draw a probability tree for each of the above experiments.
   b. In which experiment, One or Two, is the event of obtaining a blue marble on the second draw independent of the event of obtaining a blue marble on the first draw? One

The tree diagrams for Experiments One & Two have the same basic structure:
The first marble drawn might be blue ("B") or White ("W"), so, keeping it simple, we branch two ways.
The second marble might be B or W also, so the tree diagram again branches two ways at each end.
Moreover, the probabilities that belong to each of the first two branches are the same in both diagrams.
But what is different about the two diagrams should be the probabilities we assign to the second branches.

In experiment One, the chance of getting a blue marble the second time is completely unaffected by (and independent of) the color of the first marble.
In experiment Two, where the first marble is not replaced in the jar, the color of the first marble does affect the chance of getting a blue marble (or white) the second time.

We call this changed probability "the conditional probability".

> The CONDITIONAL PROBABILITY of B given A, written P(B|A), is: $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$ ❖
> (provided P(A) is NOT 0)

Multiply both sides of this by P(A), and this formula leads to the

> General Multiplication Rule for probability: $P(A \text{ and } B) = P(A) \cdot P(B|A)$.

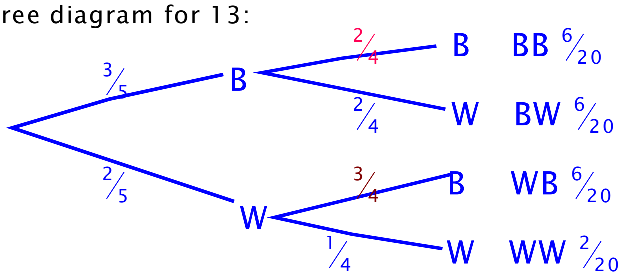
(This formula states what we knew intuitively in making the probability tree for #13.)
The General Multiplication "Rule" can be used in all cases, whether the events involved are dependent or independent. (That's why it's called "general", whereas the first multiplication rule we saw– for independent events– applies only in special circumstances!)

Conditional probability may be computed using the formula ❖ above, but in many cases we find the conditional probability intuitively.

Quite often, conditional probability is easier to find than the "straight" probability.
For example, in #13 we easily, intuitively, found the right values to place on our tree diagram:

P (B on the 2nd try | B on the 1st try) = $2/4$ and P(B on the 2nd try | W on the 1st try) = $3/4$.

The tree diagram for 13:



The most-missed question was #13b*.
Most students answered with the conditional probability shown in red above.
But that was NOT the question!
P(B on 2nd try) is not the same as P (B on the 2nd try | B on the 1st try) !

P(B on 2nd try) = P(BB or WB) = P(BB) + P(WB) = $6/20 + 6/20 = 12/20 = 3/5$

One final comment on this: the conditional probabilities are easy to find, as you build the tree diagram, because you are thinking about the experiment, picturing it.

The straight probability, without conditions, is more difficult to find (for most students).

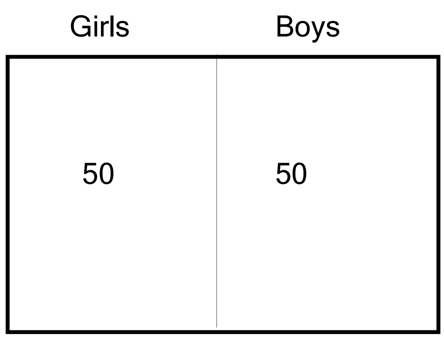
15. Here is another example which may show that the definition of conditional probability makes sense.

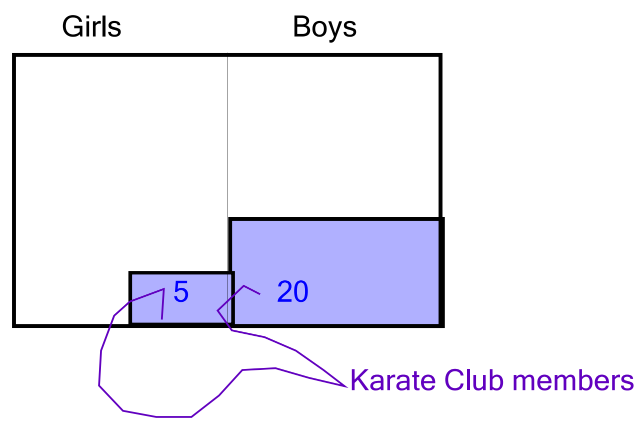
Suppose at Unity Community High School:

> HALF of the 100 students are girls; HALF are boys.
> Of the girls, 5 belong to the Karate Club, 45 do not.
> Of the boys, 20 belong to the Karate Club, 30 do not.

a. If we select a student at random from UCHS, what is the chance that we have a girl ?

b. Now: suppose a student was selected at random from UCHS, and you can see that student happens to belong to the Karate Club (you can tell by the Karate outfit). You are then asked to bet on whether the student is a girl or a boy. Would you want to bet that the student is a girl? No What is the chance that student is a girl? P(Girl|Karate) = $^5/_{25}$





If the student selected is in the Karate Club, doesn't it seem more likely the student is a boy than a girl, since most of the Karate Club members are boys? (In the Karate Club, Boys outnumber the Girls, 4 to 1 !)

Here is what we get using the definition of conditional probability:

$$P(\text{ Girl} \mid \text{Karate }) = \frac{P(\text{ Girl \& Karate })}{P(\text{ Karate })} = \frac{5/100}{25/100} = \frac{5}{25}$$

Knowing the student is in Karate, there are ONLY 25 students possible, and there is slim chance the student is a Girl, since there are SO FEW Girls in Karate ...only 5 !)

Conditional Probability asks for the probability of an event under special circumstances... Circumstances which effectively narrow the Sample Space.
For another example, see #18 on the next page.

P.S. Here are some probabilities regarding the above example:

P(G) = ½

P(B) = ½

P(K) = $^{25}/_{100}$

P(G and K) = $^5/_{100}$

P(B and K) = $^{20}/_{100}$

P(G | K) = $^5/_{25}$

P(B | K) = $^{20}/_{25}$

If events A and B are independent, then: $P(B|A) = P(B)$ and $P(A|B) = P(A)$

so the general multiplication rule $P(A \text{ and } B) = P(A) \cdot P(B|A)$
becomes the special rule for independent events: $P(A \text{ and } B) = P(A) \cdot P(B)$

The last formula is the special-case rule we saw above.

16. Can this special-case rule be used to find the probability of obtaining two blue marbles in experiment 12? Yes 13? No; in 13 we must use conditional probabilities.

17. Can the general rule be used to find the probabilities discussed in example 11? Yes

18. All students at Maxwell High School take one elective; one-third of the seniors take art as their elective. One-fifth of all juniors, sophomores and freshmen take art. Assuming 25% of the students are seniors, 25% juniors, 25% sophomores, 25% freshmen, what is the probability the student is taking art as an elective given the student was selected at random ...

a. from the senior class at Maxwell H.S.? $\frac{1}{3}$
b. from the Maxwell H.S. student body? $\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{4}$
c. from the Maxwell H.S. student body, IF we know that student is a senior? $\frac{1}{3}$

Question 18c is virtually identical to 18a. If the student has been selected at random from the entire student body (in which case the SS is the entire student body), but then we are supplied with (given) additional information about the student (in this case the fact that the student is a senior), the additional information effectively reduces the SS to that subset for which the additional condition is true. That is, although the student was selected from the entire student body, knowing the student is a senior is tantamount to having selected the student from among the seniors!

19. If A & B are events in a sample space, & $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, & $P(A \text{ and } B) = \frac{1}{4}$, $P(B|A) = ?$

By conditional probability: $P(B|A) = P(A \text{ and } B) = \dfrac{P(A \text{ and } B)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

20. Are the events A and B in #19 independent? How do you know? (Two ways!)

No, P(B|A) is not the same as P(B). So A's occurrence affects B's probability.
$P(A \text{ and } B) \neq P(A) \cdot P(B)$.

21. A jar contains five marbles: one green, two red and two blue. If three marbles are drawn from the jar, what is the probability that they are GRB– in that order?

$P(GRB) = P(G) \cdot P(R|G\ 1^{st}) \cdot P(B|\ G\ 1^{st}\ \&\ R\ 2^{nd}) = \frac{1}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{15}$

22.☆ If three marbles are drawn from the jar in #21, what is the probability of getting one of each color? (Hint: do they have to come out in the order GRB?)

There are six ways to get one of each. And each has the same probability, for instance:
$P(RBG) = \frac{2}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{15}$ [Notice this is the same result as P(GRB).]

So P(one of each) = P(GRB) + P(RBG) etc. = $\frac{6}{15} = \frac{2}{5}$

If two teams of evenly-matched players are playing a game, but one team has 2 players, and the other has 3.... You might describe the game as "unfair" because "it's 3 against 2".

1. In a situation closer to probability experiments: suppose you are asked to draw a marble from a jar containing 3 red and 2 white marbles. The catch is if the marble is red, your opponent wins $1 from you, but if the marble is white, you win $1. You might say this game is unfair! And you are right, because it's 3 against 2! That is, of the five equally likely outcomes (marbles) in the jar, 3 of them favor your opponent, and only 2 favor you. We can express this inequity by saying:

THE ODDS ARE AGAINST YOU, 3 to 2. ...or: The odds against you are 3:2.

... We can also say that: The odds in your favor are 2:3. (This statement not as intuitive!)

A. What are the "odds of red" if there are 30 red & 20 white marbles*?
(*all other conditions remaining the same)
"Odds" compares "for" and "against"– "odds" are ratios.

(The proportions matter!) 30 : 20 against you.... or we can say 3 :2 against you.

B. What is the probability you will win? 2/5 (Probability ≠ Odds!)

What is the probability you will lose? 3/5

Now consider the following somewhat abstract question:

2. The odds in favor of Our School winning the Big Game are 5 : 3.

A. What are the odds against us? If the odds in our favor are 5 : 3, then odds against us are 3 : 5

B. What is the probability that Our School will win the Big Game? 5/8 Probability compares Favorable outcomes to ALL

One way to find the right answer quickly and confidently is to PICTURE THE ODDS– as if they are "marbles in the jar" (or equitable outcomes in the sample space).
Since the odds in Our Favor are 5 to 3, it is as if there are 8 outcomes– 8 marbles in the jar... and 5 of them are "winners" and 3 are "losers". Then we can "see" that the probability of winning is 5 out of 8, or 5/8.

Potential outcomes: ☺ ☺ ☺ ☺ ☺ ☹ ☹ ☹

ODDS compares Favorable outcomes to Unfavorable.
... "For" vs. "Against"

3. One of the difficulties we encounter in "odds" questions arises from awkward terminology: Suppose there are 7 unfavorable outcomes and only 3 favorable outcomes and in a sample space with 10 equally likely outcomes. Are the odds in our favor ? No, they are against us.

But we might still be asked the question: "What are the odds in our favor?"

Answer more easily & comfortably by posing a more natural question: Are the odds in my favor?
If they tell me it's 3 for me and 7 against me,
then I naturally say the odds are not in my favor– they are against me 7 to 3.
And... if the odds against me are 7:3, then I know the odds in my favor must be 3:7.

4. You must also be able to convert from probability to odds.

If your chance of winning is 2/5, then the odds in your favor are.... Picture it !!

2 : 3

5. What are the "odds of rolling doubles" when two fair ordinary dice are rolled?

Doubles constitute 6 of the 36 equally likely outcomes in the SS.
Thus there are 6 "for" us and 30 "against" us.
So the odds ARE AGAINST rolling doubles 30 : 6, which tells us
the ODDS in favor of rolling doubles are 6 : 30, or 1 : 5