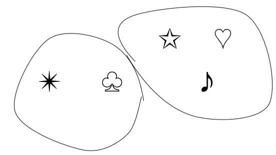
USE SMALL NUMBERS (so that the total does not exceed 10, or perhaps 20) to introduce the concept of addition. MODEL ADDITION using both SETS of objects and MEASURE (may be done on the Number Line).

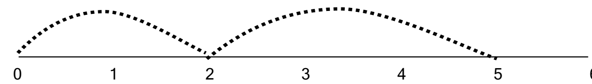
SET MODEL (Mathematicians define addition of whole numbers, such as “2+3", using SETS and COUNTING.)

E.g. To determine the sum of 2 and 3, let A = {☆,✳} and B = {♧,♪,♡}.
2 + 3 = n(A∪B) = n({☆,✳,♧,♪,♡}) = 5.

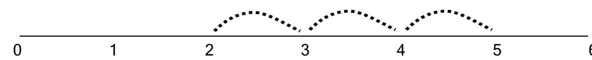


MEASUREMENT MODEL
We also introduce students to measurement models of addition:

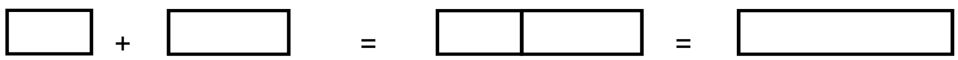
E.g. 2+3 on the number line can be shown:



Or



E.g. 2+3 using cuisenaire rods:



Properties of addition (+) on the Whole numbers (W):

CLOSURE: For any two whole numbers, a & b, a + b is a whole number. Eg 2+3 = 5

EX: + on the set of EVEN numbers { 0, 2, 4, 6, 8, 10, 12, ··· }

EX: + on the set of ODD numbers { 1, 3, 5, 7, 9, 11, 13,··· }

COMMUTATIVITY: For any two whole numbers, a and b, a+b = b+a. Eg 2+3 = 3+2
5 = 5

ASSOCIATIVITY: For any a, b and c in W: (a+b) + c = a + (b + c) Eg (2+3)+1 = 2+(3+1)
5 + 1 = 2 + 4
6 = 6

“ANY ORDER PROPERTY” is the result of COMMUTATIVITY & ASSOCIATIVITY.

EG 9 + 7 + 1 + 3 = 9+1 + 7+3 = 10 + 10 = 20
((9 + 7)+ 1)+ 3 = (9 + (7+1)) +3 = (9 + (1+7)) + 3 = ((9+1) +7) + 3) = (9+1) + (7+3) = 10 + 10 = 20

TEACHING ADDITION FACTS:

Children need to learn addition facts from 1+1 to 9+9.

0+n = n is the IDENTITY property – which should be brought to attention

n+1 = the next number. n+2 is almost as easy.

Counting forward (illustrate on the number line) Memorize Sneaky tricks:

WHAT PROPERTY justifies: (7+8)+2 = 7+(8+2) ? (2+5) + 8 = (5+2) + 8?

((7+ 22)+73 = 22+(7+73) ?

((83 + 22) + (17+ 44) ?

8742 + 0 = 8742

(Note we are using COMPENSATION to make addition easier in many of the above cases)

Addition mastery forms a firm foundation for the other arithmetic operations. The other operations are understood well only with a good grasp of the concept of addition as well as the “addition number facts”. For instance,

SUBTRACTION of Whole numbers.is DEFINED IN VIA ADDITION.... If $a < b$ then $b - a = c$, where $a + c = b$.
EG $5 - 3 =$, because $3 +$ $= 5$
MISSING ADDEND ↗

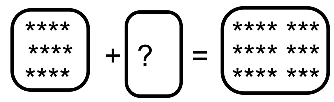
TERMINOLOGY: MINUEND – SUBTRAHEND = DIFFERENCE

But in early grades, it is usually defined as “take-away”....

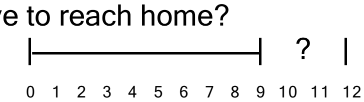
3 INTERPRETATIONS OF SUBTRACTION: PART-WHOLE, TAKE-AWAY, COMPARISON

PART-WHOLE: closely matches the “missing addend” definition:

PART of a set or quantity is given; WE ASK HOW MUCH TO MAKE IT WHOLE.
Set model: of the 21 students in a class, 12 are girls; how many are boys?

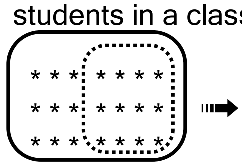


Measure model: Chris lives 12 miles from his work. After driving 9 miles, Chris stops for dinner. How many more miles must he drive to reach home?

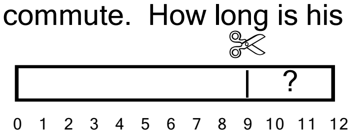


TAKE-AWAY: PART of a set or quantity is REMOVED; WE ASK HOW MUCH IS LEFT.

Set model: There were 21 students in a class, and 12 of them left the room. How many are left?



Measure model: Chris used to live 12 miles from work, but he just moved closer, cutting 9 miles off his commute. How long is his commute now?

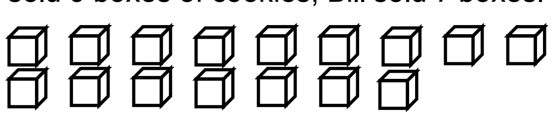


OR....

COMPARISON: COMPARE two sets, or two quantities, and ASK FOR THE DIFFERENCE.
(DIFFERENCE = larger – smaller.)

Set model: Amy sold 9 boxes of cookies, Bill sold 7 boxes. How many more did Amy sell than Bill?

AMY
BILL



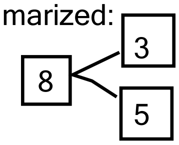
Measurement model: Amy is 48" tall, and Bill is 60" tall? What is the difference in their heights?

Thinking Strategies for subtraction:

Emphasize the "Four Fact Families"

EG 5 + 3 = 8   8 – 5 = 3
3 + 5 = 8   8 – 3 = 5

Summarized:



The Number Bond

Counting down: 25 – 8 = (25 – 5) – 3 more ______________________

Counting UP: 25 – 17 = ______________________

COMPENSATION (unlike addition compensation)... Do the same thing to BOTH

Reminder– Addition compensation: 97 + 125 = 100 + 122 = . . .

Subtraction compensation: 125 – 97 = 125+3 – (97+3) = 128 – 100 ... making the subtrahend NICE.

98 – 82 = 96 – 80 We increment or decrease BOTH.

SUBTRACTION– [LACK OF] PROPERTIES

CLOSURE in early grades we ask students to find the difference, which eliminates this problem, but we have to make sure we don't give a problem such as 2 – 5.

COMMUTATIVITY Hey, 5 – 3 is 2 but 3 – 5 is nonsense in early grades, and – 2 later. Ask your BANK!

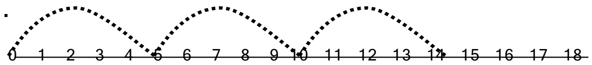
ASSOCIATIIVITY (8 – 3) – 2 $\neq$ 8 – (3– 2)

IDENTITY – zero is NOT an identity, because it does not work "on both sides": 5– 0 = 5 but 0 – 5 is not 5.

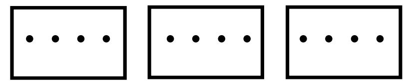
Definition: a·b = b+ b+ ··· + b (a of them).
That is, multiplication may be defined as repeated addition.

EG: 3·5 = 5+5+5

Visualization:



EG: 3·4 = 4+4+4 = number of objects in

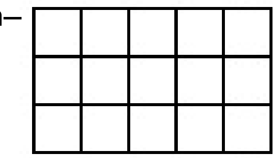


(3 groups of 4)

But mathematicians say a·b = the number of ordered pairs where the 1st item is chosen from a set of a items, and the 2nd from a set of b items.
EG: 3·5 = the number of pairs in this table...

	d	e	f	g	h
a	a,d	a,e	a,f	a,g	a,h
b	b,d	b,e	b,f	b,g	b,h
c	c,d	c,e	c,f	c,g	c,h

It is certainly appropriate to introduce young students to this visualization of multiplication– 3·5 is the number of square units in a rectangle that is 3 units by 5 units.

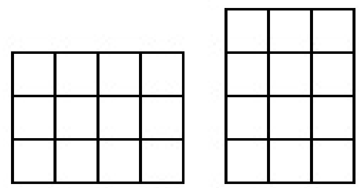


Properties of multiplication:

Closure: If a and b are Whole numbers, then a·b is a Whole number.

Commutative: For any whole numbers, a and b, a · b = b · a

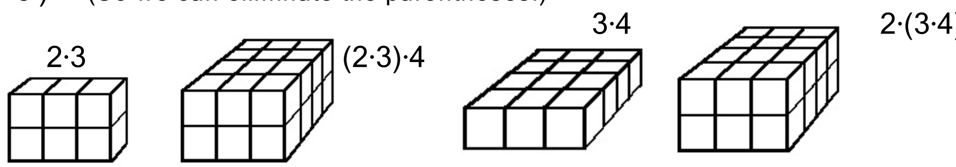
EG 3x4 = 4x3 as illustrated at right....



3 · 4 = 4 · 3

Associative: (a · b) · c = a · (b · c) (So we can eliminate the parentheses.)

EG: (2·3)·4 = 2·(3·4)
6 · 4 = 2·12
24 = 24



Together, the commutative and associative properties justify the “Any order” property for multiplication.

EG: (25·38)·4 = Why? (25·38)·4 = (38·25)·4 = 38·(25·4) = 38·(100) = 3800

Compensation: To multiply a number by 5, multiply by 10 and take half... or take half and multiply by 10

EG 26 · 5 = (13·2) · 5 = 13 · 10.

Identity: 1 is the multiplicative identity element. a·1 = 1·a = a for any whole number
Annihilator: Multiplying any number by 0 yields 0. 0 is the Additive Identity, but is the Multiplicative Annihilator!

Distributive Property of Multiplication over Addition*: a · (b + c) = a · b + a · c

(b + c) · a = b · a + c · a

* or Subtraction !

Just replace the + with - !

EG 2 · (4 + 3) = 2 · 4 + 2 · 3

♥♥♥♥♦♦♦
♥♥♥♥♦♦♦ = ♥♥♥♥ ♥♥♥♥ + ♦♦♦ ♦♦♦

EG 3 · (52) = 3 · (50 + 2) = 3 ·50 + 3 · 2 = 150 + 6 = 156

52
x 3

Mental Math: Multiply by 9 (use 9 = 10 – 1) ... by 99 (use 99 = 100 – 1) etc.
Multiply by 11 (use 11 = 10 + 1) ... by 101 (use 101 = 100 + 1) etc.

EG 49 · (101) = 49 ·100 + 49 · 1 = ···

EG 12 · 97 = 12 · (100 – 3) = ···

Before we start on division, a few additional thoughts on the distributive property: $2 \cdot (4 + 3) = 2 \cdot 4 + 2 \cdot 3$

[Parentheses are not needed in the *expression on the right*, due to an accepted convention– that multiplication takes precedence over addition (i.e. happens first). So 4x2 + 4x3 **means** (4x2) + (4x3).

Notice if we did not have such a convention, expressions mixing + and x could be ambiguous: 5 + 3 x 2 = 5 + 6 = 11
... Without the convention this could be interpreted as: 5 + 3 x 2 = 8 x 2 = 16. ← Incorrect under P E M D A S]

Note **Addition does NOT distribute over multiplication**:

Is 5 + (2 x 3) the same as (5 + 2) x (5 + 3)? [Check it out!]

DIVISION

Just as we define subtraction in terms of* addition, we define division in terms of* multiplication. *i.e. as "reverse" or "opposite" of

For a & b in W: IF a x c = b, then the QUOTIENT* b ÷ a = c . (* also written $\frac{b}{a}$ ...only in higher grades)
b is the DIVIDEND & a is the DIVISOR.

E.g. 6 ÷ 2 = 3 since 2 x 3 = 6 .
But 5 ÷ 2 is undefined in W, as there is no c in W such that 2·c = 5.

Properties (?) of Division:

1. ~~CLOSURE~~: Division is NOT closed on W. Only for certain pairs is b ÷ a in W.

E.g. 6 ÷ 2 = 3, but 5 ÷ 2 is not in W, nor is 2 ÷ 6, nor is 3 ÷ 2 and so on.

2. ~~COMMUTATIVITY~~: Division is NOT commutative. a ÷ b = b ÷ a only when |a| = |b|.

E.g. 6 ÷ 2 = 3; but 2 ÷ 6 is not defined in W (& when our U expands to handle 2÷6, it won't be 3!)

3. ~~ASSOCIATIVITY~~: Division is NOT associative. (a ÷ b) ÷ c is almost never a ÷ (b ÷ c). In fact, often one will be defined and the other not defined.

E.g. (12 ÷ 2) ÷ 2 = 6 ÷ 2 = 3 ; but 12 ÷ (2 ÷ 2) = 12 ÷ 1 = 12 .

4. ~~IDENTITY~~: NO division identity. "one-sided" identity: a ÷ 1 = a for all a in W. But 1 ÷ a is not a!!

5. NEVER by 0: a ÷ 0 is never defined. If a ÷ 0 = b then a = 0 x b = 0 .

At this point, you might think 0 / 0 is OK... but it is not, because 0 / 0 = 42978 (today only).
(HUH? On, sorry, that was Tuesday of last week! This week 0/0 = 19π... next week, ???)

[Zero is the multiplicative ANNIHILATOR- in multiplication, 0 "turns" everything into 0– therefore, it is not surprising we don't have an inverse (division) for 0.]

Properties of Division and Addition/Subtraction:

~~DISTRIBUTIVE~~: division does NOT distribute over addition or subtraction.
A one-sided, almost-distributive property: When division "works" for all parts: (b ± c) ÷ a = b÷a ± c÷a.

E.g. Does (15 – 10) ÷ 5 equal 15÷5 – 10÷5 ?
5 ÷ 5 3 – 2
1 1

On the other hand:
(13 + 2) ÷ 5 = 3 , while 13÷5 + 2÷5 can't be computed until our universe includes fractions.
(But 13/5 + 2/5 = 15/5, so we will have one-sided distributive property.)

But the "left-sided" distributive property fails completely:
Does 24 ÷ (4+8) equal 24 ÷ 4 + 24 ÷ 8 ?
= 24 ÷ 12 = 6 + 3
= 2 = 9 ...So 24 ÷ (4+8) ≠ 24 ÷ 4 + 24 ÷ 8

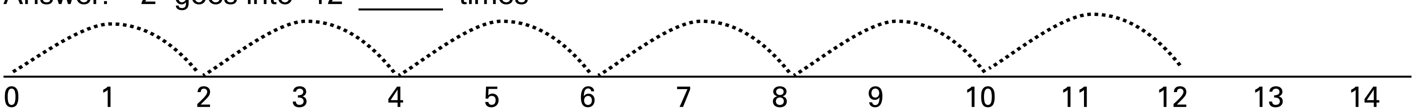
Interpretations of division:
We defined division as the "reverse" (or inverse) of multiplication. For instance, to compute 45 ÷ 15 , we seek the answer to:

$$15 \times \underline{\ ?\ } = 45$$

This is also known as the "missing factor" technique, and is certainly how we "do" division in general.

Just as multiplication can be viewed as repeated addition, division can be viewed as repeated subtraction- the "gzinta" (goes in to) process. This is another view of the MEASUREMENT model.

We might ask: “How many groups of 2 are contained in 12?”
Answer: 2 “goes into” 12 ______ times



Answer: 12 is ___ groups of 2.

Division would have given us this result: 12 ÷ 2 = 6

Another view of Division— Partition (or PARTITIVE model):

12, broken into two groups, results in what size groups? (Look at the second line below.)

TWELVE DIVIDED INTO EQUAL GROUPS:

One	♥♥♥♥♥♥♥♥♥♥♥♥
Two	♥♥♥♥♥♥ ♥♥♥♥♥♥
Three	♥♥♥♥ ♥♥♥♥ ♥♥♥♥
Four	♥♥♥ ♥♥♥ ♥♥♥ ♥♥♥
Six	♥♥ ♥♥ ♥♥ ♥♥ ♥♥ ♥♥
Twelve	♥ ♥ ♥ ♥ ♥ ♥ ♥ ♥ ♥ ♥ ♥ ♥

TWELVE WAITING TO BE PARTITIONED:

☺
☺ ☺ ☺
☺ ☺ ☺
☺ ☺
☺ ☺ ☺

Summary:

MEASUREMENT: you know the size of the group, then how many groups?

PARTITIVE: you know how many groups, then what is the size of the groups?

Teaching strategies:

(1) Learning multiplication first (both concept & facts) is key

(2) Four-Fact Families reinforce the number facts

EG $3 \times 4 = 12$ $12 \div 3 = 4$

$4 \times 3 = 12$ $12 \div 4 = 3$

(3) Division applies to problems in which some number of items is being broken into groups.

If we know the size of the groups, and are being asked “How many groups?” - measurement model
If we know how many groups, and are asked to find the number of objects ... - partitive model
Which model applies to each of these?
(a) How many 3-foot lengths of ribbon can be cut from a spool containing 45 feet?
(b) I made two dozen cookies. How many packages of 4 cookies can I make?
(c) I made 36 mini-cupcakes for my party. There are 12 guests at my party.
How many mini-cupcakes can each guest eat without taking more than a fair share?
(d) How many $3 storybooks can I buy with a budget of $39?
(e) Six place settings of china weigh 72 pounds. How much does each place setting weigh?
ANSWERS: (a) mm (b) mm (c) pm (d) mm (e) pm