

The sets **N** (natural numbers) and **W** (whole numbers) are closed under the operations of $+$ and \times . (In other words, in both **N** and **W**, we can add and multiply any two numbers, knowing the result will be found within that same universe.) However, we have seen the operation of $-$ is not closed:

SUBTRACTION ON **W** IS NOT CLOSED.

For instance, we can solve $6 - 2 = \underline{\quad}$ in **W**, because $2 + \underline{\quad} = 6$
and $6 - 6 = \underline{\quad}$ (because $6 + \underline{\quad} = 6$).

But the similar problem $6 - 7 = \underline{\quad}$ has no result in **W** because $7 + \underline{x} = 6$ has no solution x in **W**. ... 6 is one short of 7, and adding any whole number to 7 will go in the wrong direction.

In order to be able to quantify the idea of "one short", and provide **closure** for subtraction, we invent the **negative numbers**. Not surprisingly, the key to inventing them is to describe them via addition....

DEFINITIONS

- ◆ If $n \in \mathbf{N}$, then " $-n$ " is defined to be **the unique number such that $n + -n = -n + n = 0$** .
(This makes $-n$ the additive inverse of n .)
This number " $-n$ " is called "**NEGATIVE n** "; **the set of negative integers (\mathbf{Z}^-)** contains all such numbers.

- ◆ The set of **INTEGERS (\mathbf{Z})** is the union of **W** and **\mathbf{Z}^-** .

Note: **N** and **W** are subsets of **Z**. $\mathbf{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$.

We will sometimes refer to **natural numbers** as **POSITIVE INTEGERS**. ($\mathbf{Z}^+ = \mathbf{N}$).

We may think of **Z** as the union of the disjoint sets: $\mathbf{Z} = \mathbf{Z}^- \cup \{0\} \cup \mathbf{Z}^+$.

Fact: If x is any integer, there is a unique integer y such that $x + y = 0$.

If $x \in \mathbf{N}$, then $y = -n$, a negative integer. E.g. if $x = 10$, then $y = (-10)$. If $x = 0$, then $y = 0$.

If x is a negative integer and so is expressible as " $-n$ ", and then $y = n$, a positive integer.

E.g. if $x = (-2)$, then $y = 2$ since $(-2) + 2 = 0$.

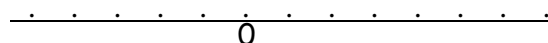
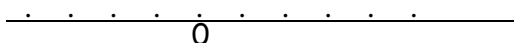
- ◆ For any a in **Z**, the **ADDITIVE INVERSE** of a , denoted " $-a$ ", is the unique number such that $a + -a = 0$. Some call this "the opposite of a ".

The additive inverse ... of 5 is $\underline{\quad}$... of 0 is $\underline{\quad}$... of -3 is $\underline{\quad}$.

⇒ Notice that " $-a$ " is not necessarily negative! ...and that $-(-a) = a$.

Illustrate 3 and -3 :

the additive inverse of -5 is:

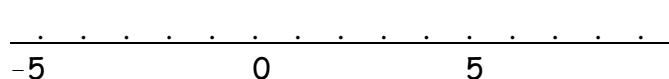


Notice the "length" of -3 is the same as that of 3.

- ◆ The **ABSOLUTE VALUE** (denoted $|a|$) of an integer a ... is:

$$|a| = \begin{cases} a & \text{if } a \text{ is } 0 \text{ or a natural number (positive)} \\ -a & \text{if } a \text{ is a negative integer.} \end{cases}$$

E.g. $|5| = \underline{\quad}$ $|-5| = \underline{\quad}$



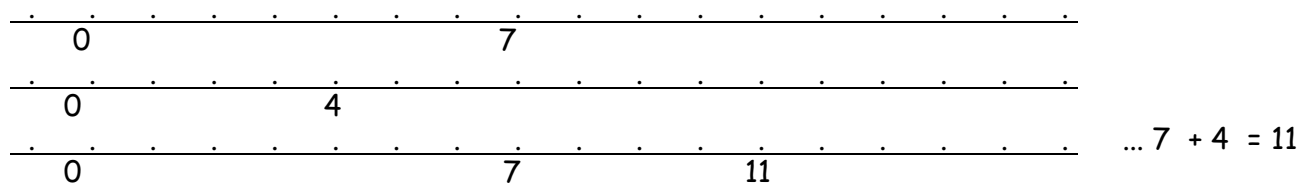
Notice the absolute value of an integer is the "size" of the number; it is also the number's "distance from 0".

Notice that :

- ◆ If " b " is a negative integer, then $|b|$ is $-b$. Since " b " is negative, " $-b$ " is the opposite— positive!

- ◆ $\sqrt{x^2}$ is not x ... it is $|x|$.

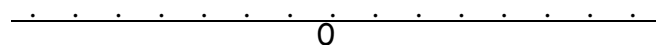
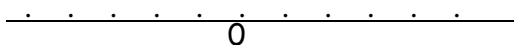
Integers are directed numbers ($-a$ is the opposite of a),. Operations with these numbers may make more sense if we view the integers as vectors (loosely speaking, as "arrows" which cannot be turned).



Since they are new, we can define addition of negative integers as we wish. But we wish consistency! The above model provides a framework to see how we should proceed.

Illustrate the sum $2 + (-3)$.

Illustrate the sum $(-3) + (-2)$.



We make the following ♦DEFINITIONS*:

*(The sum of two integers is already defined if they are both whole numbers.

Thus we need only define the sum in the case where one or both the integers are negative.)

- ♦ The sum of two negative integers, $a + b$, is $-(n + m)$ where $a = -n$ and $b = -m$.
 If a is negative and b is positive, and $|a| < |b|$, then $a + b = b + a = |b| - |a|$.
 If a is negative and b is positive, and $|a| > |b|$, then $a + b = b + a = -(|a| - |b|)$. ♦

In other words:

To add two negatives, find the sum of the corresponding positives, then take the opposite.

E.g. $-5 + -3 = \underline{\hspace{2cm}}$ $-14 + -36 = \underline{\hspace{2cm}}$

To obtain the sum of two opposite-sign numbers, we find the difference between the absolute values- between the corresponding positives. The sum has the sign of the larger size number.)

E.g. $-3 + 5 = \underline{\hspace{2cm}}$ $-34 + 20 = \underline{\hspace{2cm}}$

Properties of addition on \mathbb{Z} :

- ♦ **CLOSURE** We have defined $a+b$ for any two integers, a & b !
- ♦ **COMMUTATIVITY** $a+b = b+a$ still true. To prove it, we have to review all the cases.
- ♦ **ASSOCIATIVITY** $(a+b)+c = a+(b+c)$ likewise (& there are more cases to consider).
- ♦ **IDENTITY** 0 still works, for ALL elements $b \in \mathbb{Z}$
- ♦ **INVERSE** For each $a \in \mathbb{Z}$, there is a unique $c \in \mathbb{Z}$ such that $a + c = 0$ and $c + a = 0$.
[see above!]

COMPARING INTEGERS

Sometimes confusion arises in **comparing negative numbers**. E.g. $-5 ? -3$. (Is $?$ $>$, $<$, or $=$?)

Recall the definition of $<$ in \mathbb{W} : $a < b$ iff there is positive whole number c such that $a + c = b$.

This definition can be extended to include negative integers. E.g. $-5 + 2 = -3$, so -5 is less than -3 .

A positive amount must be added to -5 to reach -3 on the number line.

(Note also that -5 lies to the left of -3 on the (horizontal) number line.)

⇐ Some find a "vertical" number line model more useful than the typical horizontal model.

(-5 lies below -3 .)

Subtraction of whole numbers was defined in terms of addition. We see above that addition of negative and positive integers is defined in terms of subtraction of natural numbers. The following definition leads to results consistent with our previous understanding of subtraction (and addition).

♦ For $a, b \in \mathbb{Z}$, $a - b$ is the unique number c such that $b + c = a$.

Note such a number must exist, since:

if $b + x = a$, addition of $(-b)$ to both sides gives us $(-b) + (b + x) = (-b) + a$...so $x = (-b) + a$

Thus, **SUBTRACTION is CLOSED on \mathbb{Z}** .

Fact: If a and b are integers, $a - b = a + (-b)$.

Thus we can think of subtraction as addition of the inverse. For example:

$$7 - 4 = \underline{\quad}$$

$$12 - (-8) = 12 + (-(-8)) = \underline{\quad}$$

$$7 + (-4) = +(|7| - |4|) = \underline{\quad}$$

$$-4 - 17 = -4 + \underline{\quad} = \underline{\quad}$$

$$(\text{from the definition of addition of opposite-sign numbers}). \quad -4 - (-6) = -4 + \underline{\quad} = \underline{\quad}$$

The failure of some properties in W guarantees failure in \mathbb{Z} , since $W \subseteq \mathbb{Z}$. For example, there are a and b in W where $a - b \neq b - a$. In \mathbb{Z} , those results are still unequal (of course!). Therefore, subtraction will never be commutative, no matter how our universe grows. Similarly, subtraction fails to have associative and identity properties.

Properties of subtraction on \mathbb{Z} :

- ✧ **CLOSURE** See above; we finally have an answer for each subtraction.
- ✧ **COMMUTATIVITY** $7-3$ was not equal to $3-7$ before, and never will be.
- ✧ **ASSOCIATIVITY** $(7-5)-2$ was not equal to $7-(5-2)$ before, and never will be.
- ✧ **IDENTITY** 0 works only on one side.
- ✧ **INVERSE** If there is no identity, the question of inverses does not even arise.

APPLICATIONS OF INTEGER ADDITION AND SUBTRACTION

- Alec tried to climb a hill. Alec climbed 15 feet up the hill, slid back 10 feet, then climbed 5 feet and slid back 6. How much upward progress did Alec make in all?
- XX stock declined 10 points on Monday, and 15 points on Tuesday, rose 32 points on Wednesday, but fell 5 points on Thursday and rose 2 points on Friday. What was XX stock's net gain/loss for the week?
- The temperature yesterday in Fairbanks was twelve degrees (F) below zero; today it is twenty-three degrees. What was the temperature change *from yesterday to today*?
- The nucleus of an atom contains positively-charged particles called protons; negatively charged particles orbit the nucleus. The positive charge of a proton and negative charge of an electron exactly balance each other, for a net charge of 0 (In other words, they are opposites, and add up to zero, just like an integer and its inverse).
 - An atom has ten protons (+) and nine electrons (-); what is the particle's "net charge"?
 - The above particle picks up three electrons (three negative charges); what is the new net charge?
 - A strongly positive ion comes by and steals two electrons away; what is the particle's net charge now?
- A year ago, Gem had \$60 in the bank and owed Visa \$5042. Now, Gem has \$2400 in the bank and owes Visa \$746. Assuming this is the extent of Gem's finances, how have they improved?
- Optional Solve: a. $|x| = 7$ b. $|x| = -5$ c. $|x| + 1 = 7$ d. $|x + 1| = 7$ e. $|2x + 1| - 2 = 5$

Short Answers: 1. $15 + (-10) + 5 + (-6) = 4$ 2. +4pts. 3. $23 - (-12) = 35^\circ$ 4a. +1 4b. -2 4c. -1

5. (now - then) = $(2400 + (-746)) - (60 + (-5042)) = 6636$ - Gem's cash position has improved by +6,636.

6a. ± 7 6b. \emptyset 6c. ± 6 6d. $\{6, -8\}$ 6e. $\{3, -4\}$

Now we must define multiplication of integers in general.

We want this definition to be consistent with multiplication of (positive) whole numbers. How shall we define, for instance: $(3) \cdot (-2)$? $(-3) \cdot (-2)$?

Consider that multiplication of whole numbers is commutative, and $3 \cdot a = a + a + a$ when a is in W .

Now if a is in \mathbb{Z} and not in W , then a is a negative integer, and thus $a = -n$ for some $n \in \mathbb{N}$

For instance, consider how to define $3 \cdot (-2)$. Note that -2 is the additive inverse of 2 .

$$0 = 3 \cdot (0) = 3 \cdot (2 + (-2)) \dots \text{should be } 3 \cdot 2 + 3 \cdot (-2) = 6 + 3 \cdot (-2)$$

$$0 = 6 + 3 \cdot (-2)$$

... So $3 \cdot (-2)$ should be the "opposite" (additive inverse) of 6 : -6 .

By similar reasoning we would want $(-3) \cdot (-2)$ to be $+6$ (i.e. 6).

$$\begin{array}{rclcl} \text{Algebraically:} & (-3)(2 + -2) & = & (-3)(2) + (-3)(-2) & \\ & 0 & = & -6 + (-3)(-2) & \Rightarrow \quad (-3)(-2) \text{ must be } 6 \end{array}$$

We can also see how to define this via the patterns:

$3 \cdot 4 = 12$	$(-3) \cdot 3 = -9$	
$3 \cdot 3 = 9$	$(-3) \cdot 2 = -6$	
$3 \cdot 2 = 6$	$(-3) \cdot 1 = -3$	
$3 \cdot 1 = 3$	$(-3) \cdot 0 = 0$	
$3 \cdot 0 = 0$	$(-3) \cdot (-1) = 3$	(here we continue the pattern.)
$3 \cdot (-1) = -3$	$(-3) \cdot (-2) = 6$	
$3 \cdot (-2) = -6$	$(-3) \cdot (-3) =$... Thus we make this ...

◆ **DEFINITION:** For $a, b \in \mathbb{W}$: [Recall $a \cdot b = n(A \times B)$ where $n(A) = a$ and $n(B) = b$ for $a, b \in \mathbb{W}$]

$$a \cdot (-b) = -(a \cdot b) \quad (\text{This covers the case of non-negative} \cdot \text{negative.})$$

$$(-b) \cdot a = -(a \cdot b) \quad (\text{This covers the case of negative} \cdot \text{non-negative.})$$

$$(-a) \cdot (-b) = a \cdot b \quad (\text{This covers the case of negative} \cdot \text{negative.})$$

Properties of Multiplication on \mathbb{Z} :

- ✧ **CLOSURE** (See above.)
- ✧ **COMMUTATIVITY** (Proof would require examination of several cases.)
- ✧ **ASSOCIATIVITY** (Proof would require examination of even more cases.)
- ✧ **IDENTITY** (1 is still the multiplicative identity.)
- ✧ **INVERSE** In \mathbb{Z} , we still cannot solve $a \cdot (?) = 1$... unless $a = 1$ or -1 .
- ✧ **ANNIHILATOR** ($0 \cdot a = 0$ complicates everything. $\times 0$ annihilates!)
- ✧ **DISTRIBUTIVE** Multiplication distributes over addition (and over subtraction).

"Sign properties of multiplication of integers"– an observation:

Multiplication of an integer by a negative integer reverses its sign.

Thus: If both integers are same sign, the product is positive.

If the integers are opposite in sign, the product is negative.

Or:

$$+ \cdot + = +$$

$$+ \cdot - = -$$

$$- \cdot + = -$$

$$- \cdot - = +$$

We define division for Z practically the same way as for W, with the same results:

◆DEFINITION For any $a, b \in \mathbb{Z}$, IF there is some $c \in \mathbb{Z}$ such that $b \cdot c = a$, then $a \div b = c$.

Properties of Division on Z:

NONE of the "standard" properties. Division by zero can never be defined; but even problems such as $2 \div 4$ and $5 \div 3$ cannot be answered in Z. DIVISION is NOT closed on Z. When the quotient exists:

If both integers are same sign, the quotient is positive; if integers are opposite in sign, the quotient is negative. (This follows from the multiplication sign properties on Z.)

Sign properties
for multiplication

$$+ \cdot + = +$$

$$+ \cdot - = -$$

$$- \cdot + = -$$

$$- \cdot - = +$$

Sign properties
for division

$$+ / + = + \text{ ...because } + \cdot + = +$$

$$+ / - = - \text{ ...because } - \cdot - = + \text{ E.g. } 6 \div -2 = -3 \text{ because } -2 \cdot \underline{\quad} = 6$$

$$- / + = - \text{ ...because } + \cdot - = -$$

$$- / - = + \text{ ...because } - \cdot + = -$$

Summary of Properties Thus Far:

Property	+ on W	+ on Z	x on W	x on Z	- on W	- on Z	÷ on W	÷ on Z
CLOSURE	✓	✓	✓	✓	NO	✓	NO	NO
COMMUTATIVITY*	✓	✓	✓	✓	NO	NO	NO	NO
ASSOCIATIVE *	✓	✓	✓	✓	NO	NO	NO	NO
IDENTITY	✓0	✓0	✓1	✓1	NO	NO	NO	NO
INVERSE	NO	✓	NO	NO	NO	NO	NO	NO
ANNIHILATOR	NO	NO	0	0	NO	NO	NO	NO
DISTRIBUTIVE*	MULTIPLICATION over addition, over subtraction							

*not generally universe-dependent

APPLICATIONS OF INTEGER MULTIPLICATION AND DIVISION

1. Alec tried to climb a hill. With each assault of the slope, Alec climbed 15 feet up the hill, then slid back 10 feet. In six such assaults on the hill, how much upward progress did Alec make in all?
2. Alice incurs a charge of 75 cents each time she uses her bank's convenient phone banking service. Last month she used the service fourteen times. What effect did this have on her bank account?
3. Alice is a financial consultant. For each referral who opens a new bank account, Alice is awarded a \$10 bonus to her checking account. Last month Alice referred six clients, who opened one new account each. However, two of them closed their accounts in less than a month, resulting in a \$15 penalty for each. (Always read the fine print!) What was the net effect on Alice's checking account?

Short Answers: 1. $6(15' + ^{-}10') = 30'$ 2. $14(^{-}75¢) = ^{-}1064¢ = ^{-}\10.64

3. $6(10) + 2(^{-}15) = 30$. The net effect was an increase of \$30 in her checking account balance.