Algebra is not just what you learned in high school. It is a much broader topic, that encompasses operations outside doing arithmetic with numbers, or even with variables represented by letters. You can find more about it at http://eom.springer.de/a/a011340.htm.

That said, we proceed with a review of familiar elementary algebra.

At least half of us are so comfortable with algebra, it is hard to force ourselves to express mathematical relationships without resorting to algebra. It is in working with word problems that we first begin to see the need for algebra. To state the relationships we see, we have used expressions such as:

$\square + 13 = 24$
$26 - ? = 10$
$4 \times \_\_\_ =$

The type of algebra we use in elementary school uses letters to represent numbers.
This is useful when...

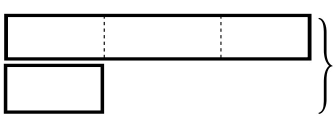
- the number is UNKNOWN (as in the above examples)
- writing IDENTITIES or general statements — eg the distributive property: $a(b + c) = ab + ac$
- expressing CALCULATIONS or RELATIONSHIPS between quantities without specifying the particular numbers
  eg P = VT [chemistry] or $A = \pi r^2$ [area of a circle]

What is an algebraic expression?

Algebraic expressions, eg: 3(x+2) and 6x - 4 ...are things that make sense
Numeric expressions, eg: 3(5+2) and 6•5 - 4 are also algebraic expressions. Each = some number.

Not algebraic expression: 14 +- 9 Nonsense, does not represent a number
Not an algebraic expression: 3x ÷ When a specific value is assumed for x, this is not a number.

Text example: There are 3 times as many boys as girls.
If there are 96 children altogether, how many girls?

Boys
Girls

96

96 ÷ 4 = 24

There are 24 girls.

Algebra solution:

\# of girls: a
\# of boys: 3a
Total 96

$$a + 3a = 96$$
$$4a = 96$$
$$a = 24$$

There are 24 girls

Text example: A farmer has 33 times as many hens as roosters.
If he has 442 chickens altogether, how many roosters?

It is now more difficult to picture using bar diagrams.

Algebra solution:

\# of roosters x
\# of hens 33x
total 442

$$x + 33x = 442$$

A plumber charges \$80 to come to the site, then \$60/hour for the time he works on a job.

If the plumber came and worked two hours, what is the charge? $C = \$80 + \frac{\$60}{\text{hr}} (2 \text{ hrs}) =$

Most texts will write this without the units of measure: $C = 80 + 60\,(2) =$

Or they will give a “formula”, also without units: $C = 80 + 60\,t$

Writing algebraic expressions (assume "a number" can be represented by "x")

eg: Express "Sum of double a number and 5" : $2x + 5$

eg: Express "Double the sum of a number and 5": $2(x+5)$

eg: Express "The product of the square of x and 3": $3x^2$

eg: Express "The square of the product of x and 3": $(3x)^2$

EG: The cost of a phone call via a certain service is \$3 per call plus \$.25 per minute for the duration.

Silvia makes a call that last five minutes. What is the cost? 4.25

Write a formula that expresses C, the cost of the call, in dollars, in terms of t, the number of minutes duration. $C = 3 + .25\,t$

We evaluate expressions by replacing the letters (also called variables) with specific numbers.

EG evaluate each of the above expressions for the value 2. 7,14,12,36,4.25,3.50

We add, subtract, multiply and divide expressions:

eg: $(2x + 5) + (3x - 4) = (2x + 3x) + (5 - 4) = 5x + 1$

(Why? The arithmetic properties apply!!!)

EQUATIONS:

An EQUATION is a statement that two expressions are equal.... has an equal sign!

Closed statement: $2x + 1 = 2x$ (NEVER true)

$2(x+3) - 2 = 2x + 4$ (ALWAYS true, no matter what value is x)

An equation that is true for all values is an IDENTITY.

Open statement: $3x - 4 = 8$ (True for some values of x, others not)

$x^2 = 16$

Formulas are open statements in which the purpose is to find one of the variables when the others are known.

EG $C = 3 + .25t$ being used to find C for particular values of t.

EG using $P = VT$ used to find P when V & T are known

...or to find T when P & V are given....

A quick word about solving equations:

Most of the equations we encounter in grade school can be solved using these basic principles:

If $A = B$ then $A + C = B + C$

and $A \cdot C = B \cdot C$ (warning: do not use the case C = 0)

and $A - C = B - C$

and $A \div C = B \div C$ (warning.... you cannot divide by 0)

importantly, all these processes are reversible.

EG

$2x + 1 = 17$ Looking at this right, you can see that x must be 8, without algebra.

$2x = 16$ We subtract 1 on both sides to get this equivalent statement.

$x = 8$ We divide both sides by 2 to get another equivalent statement, where it becomes clear that x is 8. It is important that the processes are reversible so we know the final statement is equivalent to the first.... so we know it says the same thing about x that the first statement said.

4.2 IDENTITIES

Many identities are not worthy of note. For instance one we looked at earlier:

$2(x+3) - 2 = 2x + 4$ ...is certainly not worth remembering.

The most basic identities worth knowing are the arithmetic properties.
We have given examples previously, eg $3 + 4 = 4 + 3$, $(1+2)+7 = 1+(2+7)$, et cetera. But to write the general statements, we need algebra . For example:

Commutative property of +: $a + b = b + a$ for any numbers a & b
Associative property of +: $(a + b) + c = a + (b + c)$ for any numbers a & b & c
Identity property of +: $a + 0 = a$ and $0 + a = a$ for any number a.

Commutative property of ×: $a \times b = b \times a$ for any numbers a & b
Associative property of ×: $(a \times b) \times c = a \times (b \times c)$ for any numbers a & b & c
Identity property of ×: $a \times 0 = a$ and $0 \times a = a$ for any number a.

Distributive property of × over +: $c(a + b) = ca + cb$ for any numbers a & b & c , and
$(a + b)c = ac + bc$ for any numbers a & b & c

We also express certain relationships that are useful, and worth remembering:

$\boxed{(a + b)^2 = a^2 + 2ab + b^2}$ (Which we verified in detail last week)

This identity is important because students often make the FALSE ASSUMPTION: $(a + b)^2 = a^2 + b^2$ ☹ bad !

The truth of the identity $(a + b)^2 = a^2 + 2ab + b^2$ can also be perceived geometrically:

	a	+ b
a	a·a	a·b
+ b	b·a	b·b

	10	+ 3
10	100	30
+ 3	30	9

$(10 + 3)^2 = 10^2 + 2\cdot 10\cdot 3 + 3^2$

This is especially helpful for finding things such as 11^2 & 51^2 & 1001^2 and even 99^2.

$11^2 = 10^2 + 2\cdot 10 + 1$ $51^2 = 2500 + 100 + 1$ $1001^2 = 1000^2 + 2000 + 1$

$99^2 = (100 + -1)^2 = 100^2 - 200 + 1$ (you need to know negative numbers here)

But we could also note the identity: $(a - b)^2 = a^2 - 2ab + b^2$ [can you verify this?]

Another very useful one: $(a + b)(a - b) = a^2 - b^2$ (The difference of squares)

Another tool for mental math: $41 \times 39 = (40 + 1)(40 - 1) = 40^2 - 1 =$
Here we use the easily found 40^2 to avoid $41\cdot 39$.

And it works both ways: $78^2 - 22^2 = (78 + 22)(78 - 22) = 100\,(56) =$
We do not want to have to compute 22^2 or 78^2 ,
& since we notice 78+22 = 100 (serendipity!)....

That brings us to this: $(a + b)(c + d) = ac + ad + bc + bd$ often memorized as "foil".

How do you multiply $(a + b + c)(x + y)$? $(a+b+c)(x + y) = (a+b+c)\,x + (a+b+c)\,y$
$=$

4.3 Exponents

We will use only whole number exponents for now.

DEFN If a is any number,

$a^1 = a$
$a^2 = a \cdot a$
$a^3 = a \cdot a \cdot a$
$a^4 = a \cdot a \cdot a \cdot a$
And so on....
$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{\text{n of these}}$

What if we said:

$a^1 = a$
$a^2 = a \cdot a^1$
$a^3 = a \cdot a^2$
$a^4 = a \cdot a^3$
$a^n = a \cdot a^{n-1}$

a is called the base, and n the exponent

From this definition, and not from some arbitrary rule-maker, come these facts:

EF1 $a^m \cdot a^n = a^{m+n}$ eg $x^2 \cdot x^3 = x^{2+3}$ $(x \cdot x) \cdot (x \cdot x \cdot x) = x^?$

EF2 if $m \geq n$, $\dfrac{a^m}{a^n} = a^{m-n}$ eg $\dfrac{2^5}{2^3} = \dfrac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} = 2 \cdot 2 = 2^2$

EF3 $(a^m)^n = a^{m \cdot n}$ eg $(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^? = x^{2 \cdot 3}$

EF4 $(ab)^m = a^m \cdot b^m$ eg $(2 \cdot x)^3 = (2 \cdot x)(2 \cdot x)(2 \cdot x) = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x = 2^3 x^3$

and $2^3 \cdot 5^3 = (2 \cdot 5)^3 = 10^3$

then comes a definition based on necessity:

DEFN if $a > 0$ then $a^0 = 1$

It must be so in order to be consistent with the above behavior of exponential expressions.

$a^0 \cdot a^3 = a^{0+3} = a^3$

so... a^0 must be the multiplicative identity, 1

This definition is in accord with the other facts of exponents:

$a^{m-m} = a^0 = 1$ and should not $a^m \div a^m$ be 1 ?

$(5^0)^2 = 5^{0 \cdot 2} = 5^0$ says $(w)^2 = w$

Can w be 2 ? 5?

Finally, notice we excluded a=0 in the above defnition. What IS (or could be) 0^0 ?
0^0 is undefined because it is impossible for us to make it any particular value and have that work consistently with the above observed exponential facts. To see this is so, consider that:

$a^0 = 1$ For all $a > 0$. So it seems 0^0 should be 1 to be consistent with other bases.... but

$0^3 = 0 \cdot 0 \cdot 0 = 0$ and $0^2 = 0 \cdot 0$ and $0^1 = 0$ so it seems 0^0 should be 0.

As if that were not bad enough, 0^0 would have to be 0^{1-1} which is $0 \div 0$. End of discussion!

Practice with exponents:

1. Simplify: $16^0\, 3^3\, 5^2 \div 45$
2. Solve for n: $4^n\, 24 = 6 \cdot 2^{12}$
3. Which is greater, 2^{30} or 3^{20} ?
4. Find: 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8 2^9 2^{10}

4.3A Practice with Exponents Method, Comment and Solution

1. Simplify: $16^0\, 3^3\, 5^2 \div 45$

2. Solve for n: $4^n\, 24 = 6 \cdot 2^{12}$

3. Which is greater, 2^{30} or 3^{20} ?

4. Find: $2^1\ 2^2\ 2^3\ 2^4\ 2^5\ 2^6\ 2^7\ 2^8\ 2^9\ 2^{10}$

1. $16^0\, 3^3\, 5^2 \div 45$

$= \dfrac{16^0\, 3^3\, 5^2}{45}$ Writing the division in fraction form

$= \dfrac{1 \cdot 3^3\, 5^2}{5 \cdot 3^2}$ $16^0 = 1$ and $45 = 5 \cdot 9 = 5 \cdot 3^2$

$= \dfrac{3^3}{3^2}\ \dfrac{5^2}{5}$ rearranging & restating as the product of two fractions shows we can simplify the like factors in the numerator and denominator

$= 3 \cdot 5$

$= 15$

2. $4^n\, 24 = 6 \cdot 2^{12}$ First we simplify the equation as much as possible

$4^n\, 4 = 2^{12}$ After dividing both sides by 6.

$4^n\, 2^2 = 2^{12}$ Dividing both sides by 2^2 is one way to deal with the 4.

$4^n = 2^{10}$ Resulting in a bare exponential equation.

$(2^2)^n = 2^{10}$ Expressing both sides with the same base, if possible, is the easiest way to solve an exponential equation.

Here we use $4 = 2^2$ to get the next equation.

$2^{2n} = 2^{10}$ Now we use the fact: If $2^{2n} = 2^{10}$, then the exponents must be equal.

$2n = 10$

$n = 5$

Alternately, after dividing both sides by 6:

$4^n\, 4 = 2^{12}$ says

$4^{n+1} = 2^{12}$

$(2^2)^{n+1} = 2^{12}$

$2^{2n+2} = 2^{12}$

$2n + 2 = 12$ which leads to $2n = 10$, etc as above.

3. The difficulty lies in having two different bases, and two different exponents, in our exponential forms. If we can make the bases match, or if we can make the exponents match, comparison will be easier. So how do we compare...

$2^{30}\ ?\ 3^{20}$ One thing we can see in common on both sides is the exponent 10.

$(2^3)^{10}\ ?\ (3^2)^{10}$ Still can't see the difference? We can evaluate 2^3 and 3^2...

$8^{10} < 9^{10}$ because $8 < 9$ (and $8 \cdot 8 < 9 \cdot 9$ and $8 \cdot 8 \cdot 8 < 9 \cdot 9 \cdot 9$ and so on)

4. Self-checking!. All you have to do is keep doubling, which you can do mentally. $2^{10} = 1024$.