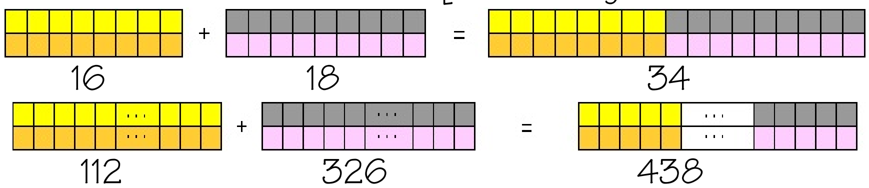
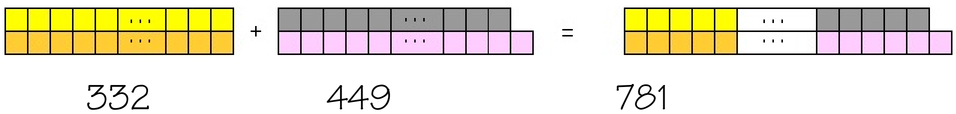
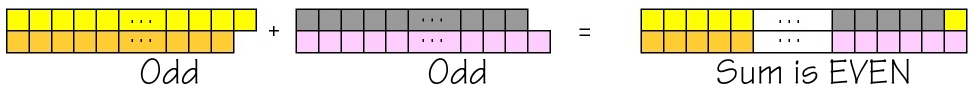
1. Illustrate 16+18 = 34 as in theorem 1.5 [i.e. showing that EVEN + EVEN = EVEN!].



2. Illustrate 332 + 449 = 781 as in theorem 1.6: EVEN + ODD = ODD



4a. Picture proof of "Odd + Odd = Even":



4b. Algebraic proof:

An odd number can be represented as even + 1: 2n + 1 for some integer n.
Another odd number can be likewise represented 2m+1 for some integer m.
The sum of two odd numbers thus can be represented as

2n + 1 + 2m + 1

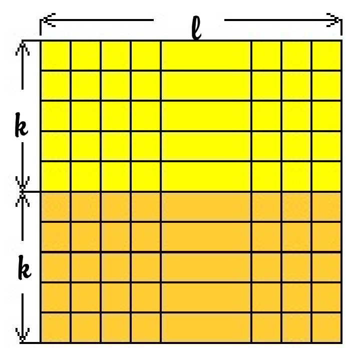
By the commutative & associative properties, this is the same as

2n + 2m + 1 + 1

which is equal to 2(n+m) + 2 by the distributive property* & 1+1 fact
and, in turn, equal to 2((m+n) + 1) by distributive property*
Which, being a multiple of 2, is EVEN.

* all references to "distributive property" on this page mean D.P. of multiplication over addition.

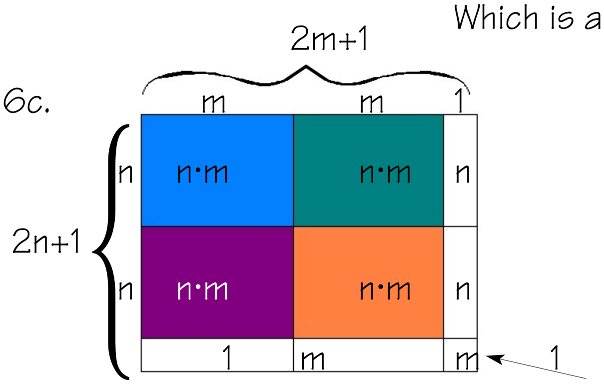
5. Quotient-Remainder Theorem states that given two whole numbers A & D, there are whole numbers Q and R such that A = Q·D + R where 0 ≤R<D.
So if A is Any whole number and D is 2, then A = Q·2 + R and 0≤R< 2
for some whole numbers Q and R. But since R is a whole number and 0≤R< 2,
R can be only 0 or 1. If R = 0, then A is a multiple of 2.
If R = 1, then A is 1 more than a multiple of 2.



6a. See far right. Note the product is even (amt. of yellow = gold)

6b. Any even number can be represented by 2k where k is integer.
Then for any whole number, ℓ, 2k·ℓ = 2· (k·ℓ) by associativity of mult.
= 2· (an integer) by closure.
Which is an even number (multiple of 2).

6c.



Each area is paired up— except the 1-unit corner !

Odd ·Odd must be Odd:
(2n+1)·(2m+1)
= (2n+1)·2m + (2n+1)·1 by distributive property*
= 2m(2n+1) + 2n+1 by commutativity & identity of ·
= 2(m(2n+1) + n) + 1 by distributive property*
= EVEN number + 1, therefore is ODD.

1. Which of the following numbers is divisible by 3? By 9? By 11?
   a. 2838: 2+8+3+8 = 21, so 2838 is divisible by 3, but not by 9.
   (...because 21 is a multiple of 3, but not of 9.)
   2 − 8 + 3 − 8 = −11, which is a multiple of 11, so 2838 is a multiple of 11.

   b. 34521: 3+4+5+2+1 = 15. So 34521 is a multiple of 3, but not of 9.
   3 −4 +5−2+1 = 3, not a multiple of 11, so 34521 is not a multiple of 11.

   c. 10234341: 1+0+2+3+4+3+4+1 = 18, so 3|21 and 9|21.
   1 −0+2−3+4−3+4−1 = 11−7 = 4 so 11 ∤ 10234341.

   d. 792: sum of digits = 7+2+9 is a multiple of 9; Alternating sum of digits = 7−9+2 = 0. } ∴ 792 is a multiple of 3, 9 and 11.

e.	8394:	3|8394	9 ∤ 8394	11 ∤ 8394
f.	26341:	3∤26341	9 ∤ 26341	11 ∤ 26341
g.	333,333:	3|333,333	9|333,333	11|333,333
h.	179:	3 ∤ 179	9 ∤ 179	11 ∤ 179

2. Which of these numbers divides 5192132? 3, 4, 5, 8, 9, 11
   Sum of digits is 2·9+5, so this number is not divisible by 3 (or by 9!).
   Last digit is not 0 or 5, so this number is not divisible by 5.
   Last two digits form the number 32, which is a multiple of 4. Therefore 4|5192132.
   Last three digits form the number 132. 8|132. So 8|5192132.
   Alternating sum of digits is 5−1+9−2+1−3+2 = 11. So 11|5192132.

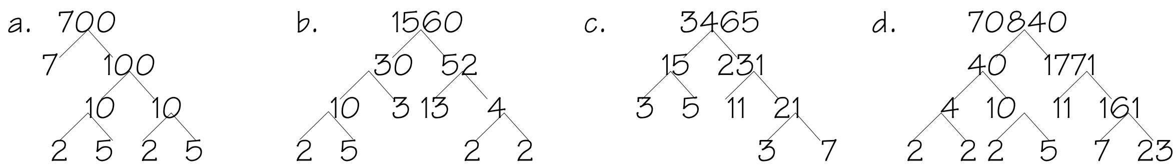
4. (Re: PT4A p25):
   Divisibility tests described on that page are for 2, 3, and 5

5. If 18|n then 3|n and 6|n.
   (If n = 18·w then n = 2·3·3·w, so n is a multiple of 2 and of 3 and of 6 and of 9.)
   The converse is not true.
   If 3|n and 6|n, it does NOT follow that 18|n.
   One counterexample is sufficient to demonstrate this.
   Consider n = 6. 3|6 and 6|6. But 18 ∤ 6.
   Consider n = 12. 3|12 and 6|12. But 18 ∤ 12.
   Consider n = 24. 3|24 and 6|24. But 18 ∤ 24.

   In general:

   If a|c and b|c it is not necessarily true that ab|c.
   HOWEVER, if a and b are relatively prime (have no prime factors in common),
   THEN if a|c and b|c , it does in fact follow that ab|c.
   For instance, if 2|n and 9|n then it follows that 18|n.
   Why this fails when a & b are not relatively prime: if a & b have some prime factor(s) in common, is due to the "overlap". E.G. 12 = 2·2·3, so having 3 as a factor, and having 6 = 2·3, does not guarantee that 3·6 = 3·2·3 is a factor (no guarantee that 3 is a repeated prime factor).

1. a. b. c. d.



$700 = 2^2\, 5^2\, 7$ $1560 = 2^3\, 3{\cdot}5{\cdot}13$ $3465 = 3^2\, 5{\cdot}7{\cdot}11$ $70840 = 2^3\, 5{\cdot}7{\cdot}11{\cdot}23$

2. Determine whether the number is prime (P), composite (C) or neither (N).

a. C 12 = 3·4, showing that 12 is C.
b. C 123 is a multiple of 3 (look at sum of digits!), thus is C.
c. C 1234 is a multiple of 2 (see the last digit!), thus is C.
d. C 12345 is a multiple of 5 (see the last digit!), thus is C.
e. C 154 is even!
f. C 102302320 is obviously a multiple of 2 (and 5 and 10) and who knows what else? !
g. N 1 is neither prime nor composite; it is a unit.
h. P 97 is not divisible by 2, 3, 5, or 7. The next prime, 11, > $\sqrt{97}$. (Put another way $11^2 > 97$.)

3. a. $63 = 9{\cdot}7 = 3{\cdot}3{\cdot}7 = 3^2{\cdot}7$ b. $768 = 4{\cdot}192 = 4{\cdot}12{\cdot}16 = 2^8{\cdot}3$
c. $324 = 4{\cdot}81 = 4{\cdot}9^2 = 2^2{\cdot}3^4$ d. $361 = 19^2$
e. $154 = 2\cdot 7\cdot 11$ f. $1024 = 2^{10}$
g. $480 = 2^5{\cdot}3{\cdot}5$ h. $10{,}000 = 10^4 = (2{\cdot}5)^4 = 2^4\cdot 5^4$

4. a. Factors of 56, in pairs: $56 = 2^3{\cdot}7$

1	2	4	7	8
56	28	14	8	7

Factors of 84, in pairs: $84 = 2^2{\cdot}3{\cdot}7$

1	2	3	4	6	7
84	47	28	21	14	12

Factors of 144, in pairs: $144 = 2^4{\cdot}3^2$

1	2	3	4	6	8	9	12
144	72	48	36	24	18	16	(-)

n has odd set of factors if and only if n is a square, so that each prime occurs an even # of times.

5. a. The prime factorization of 10! $= 1{\cdot}2{\cdot}3{\cdot}4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10$
is $2{\cdot}3{\cdot}(2{\cdot}2){\cdot}5{\cdot}(2{\cdot}3){\cdot}7{\cdot}(2{\cdot}2{\cdot}2){\cdot}(3{\cdot}3){\cdot}(2{\cdot}5) = 2^8{\cdot}3^4{\cdot}5^2{\cdot}7$

b. 10! is divisible by 10, since it is (9!)·10... ie it is a whole number multiple of 10.
10! is divisible by 30, since we see 3 and 10 in the factorization.
10! is divisible by 120 = 3·4·10 since we see 3 and 4 and 10 separately in the factorization.
10! is not divisible by 1000, since $1000 = 2^3{\cdot}5^3$ and there are only two 5-factors (ie 5^2) in 10!
Any number that divides 10! must be
the product of factors in $2^8{\cdot}3^4{\cdot}5^2{\cdot}7$, not to exceed the powers shown.

c. 30! is divisible by $2{,}400{,}000 = 2^8{\cdot}3{\cdot}5^5$
since all those powers of of primes appear in the prime factorization of 30!

d. Find the largest N for which 18! Is divisible by 12^N...
$12^N = 2^{2N}{\cdot}3^N$
18! = 1·2·3·4·5·6·7·8·9·10·11·12·13·14·15·16·17·18 has 3 as a factor 8 times, but not 9.
So the largest N for which $12^N \mid 18!$ is N = 8. (There are enough 2s– $2^{16} = (2^2)^8$ in fact.)

1. a. 127 IS PRIME. (We checked 2,3,5,7 & 11; 13 is too big because 13·13 = 169 , larger than 127.)
   b. 129 = 3·43 is not prime
   c. 327 = 3·109
   d. 221 = 13·17
   e. 337 IS PRIME (We checked 2, 3, 5, 7, 11, 13, 17; 19 is too large, since 19·19 =361)
   f. 223 IS PRIME (We need check only 2, 3, 5, 7, 11, and 13)
   h. 859 IS PRIME (It's necessary to check 2,3,5,7,11,13,17,19,23, 29; 31 is too large.)

2. The largest prime less than 1000? 999 is a multiple of 9, thus not prime.
   998 is even, a multiple of 2, not prime.
   997? Our divisibility tests for 2,3,5 and 11 eliminate those primes. We still must try 7, 13, 17, 19, 23, 29, 31; 37 is too large. None of the primes listed divides 997, so 997 must be prime.

3. a. Claim: all odd numbers are prime.
   To disprove this, we show an odd number that is not prime..
   We may present the number 1– which is a "unit" , neither prime nor composite.
   Or we may present the number 9, which is odd but not prime.
   (Or 15, or 25, or 27, or 33, et cetera. The product of any odd numbers is odd!)

   b. Claim: $(a+b)^2 = a^2 + b^2$.
   Let a be any non-zero number, say 1. Let b be any non-zero number, say 1.
   The left side then is $(1 + 1)^2 = 2^2 = 4$, while the right side is $1^2 + 1^2 = 1 + 1 = 2$. $4 \neq 2$.

   c. Claim: For each whole number n, $n^2 + n + 11$ is always prime.
   For n = 1,2,3,4,5,6,7,8,9 the values are: 13, 17, 23, 31, 41, 53, 67, 83, 101....
   But for n = 10, $n^2 + n + 11$ is $10^2 + 10 + 11 = 121$, which is 11^2 , and thus not prime.
   (By the way, look at this: If n= 11, then $n^2 + n + 11 = 11^2 + 11 + 11$ – which is clearly a multiple of 11 .

4. Prove the sum of three consecutive numbers is a multiple of 3.
   Let the first number be called "n". The consecutive numbers are then n, n+1 and n+2.
   Their sum, n + n+1 + n+2 = 3n + 3 = 3(n+1) a whole-number multiple of 3.
   (Since n is a whole number, n+1 is also.)

5. 6! = 1·2·3·4·5·6 is divisible by 2,3,4,5,6.
   a. 6! + 2 is (a multiple of 2) + 2, which is, in turn a multiple of 2.
      6! Is a multiple of 3, so 6! + 3 is (a multiple of 3) +3 = a multiple of 3.
      Similarly, 6! =4 is a multiple of 4, and 6! + 5 is a multiple of 5, and 6! + 6 is a multiple of 6.

   b. A factor of 31! + 29. 29 (because 29 | 31! and 29|29, therefore 29 divides the sum.)

   c. 5000 consecutive composite numbers:
      5001! +2 is a multiple of 2; 5001! +3 is a multiple of 3; 5001! + 4 is a multiple of 4;
      5001! +5 is a multiple of 5; 5001! +n is a multiple of n, for each n starting with n=2,
      ending with n = 5001. That's a list of 5000 consecutive numbers that are composite.

Another example demonstrating there are many ways to solve any problem:

#6. Every prime greater than 3 is odd (because the only even prime number is 2).
Therefore every prime > 3 can be written as p = 2n+1, for some whole number n.
The number before p is 2n, and the number after is 2n+2.
The multiples of 3 are three units apart.
Since 2n, 2n+1, and 2n+2 are consecutive whole numbers, at least one of these numbers must be a multiple of 3.
The multiple of 3 cannot be p, since p is prime and >3.
Therefore either 2n is a multiple of 3 or 2n+2 is.
Whichever it is, that number is divisible by both 2 and 3, therefore is a multiple of 6.
QED■.

Using the suggestion of the text:

Let p be a prime number > 3.
Then p = 6n + r where n is a whole number, and r is less than 6 (by the Quotient-Remainder Theorem).
The remainder, r, cannot be 0 or 2 or 3 or 4, since p is prime.
So r must be 1 or 5.
If r =1, then p – 1 = 6n + 1 – 1 = 6n is a multiple of 6,
so the number 1 less than p is a multiple of 6.
If r = 5, then p+1 = 6n + 5 +1 = 6n+6 = 6(n+1) is a multiple of 6,
so the number 1 more than p is a multiple of 6.
In either of the only two possible cases, the number 1 less than p or the number 1 greater than p is a multiple of 6. QED■

#7. Michelle thinks that if she makes the product of a list of primes— e.g. 2·3·7 = 42, then adds 1: 42+ 1 = 43, she will always obtain a prime number. (This is because she saw the proof that the list of primes must be infinite, which was based on a similar methodology. But keep in mind, the supposition with which that started, that the list of primes is finite, was false.)
By her technique, the next number in her list will be 2·3·7·43 + 1 = 1,807 .
This number is not prime: 1807 = 13·139 .

1a. Multiples of 15 are: 15, 30, 45, 60, 75, 90, ⋯
12 $\nmid$ 15 and 12 $\nmid$ 30 and 12 $\nmid$ 45. But 12 | 60. So 60 is a common multiple of 12 and 15.

1b. (Re PT 4A, Practice 1B, p.27 #1, #4–7):

#1. Factors of 18: 1, 2, 3, 6, 9, 18
#4 a. Factors of 8: 1, 2, 4, 8
b. Factors of 15: 1, 3, 5, 15
c. Factors of 20: 1, 2, 4, 5, 10, 20
d. Factors of 50: 1, 2, 5, 10, 25, 50
e. Factors of 75: 1, 3, 5, 15, 25, 75
f. Factors of 98: 1, 2, 7, 14, 49, 98
#5. Aside from 1, name a common factor of 15 and 6: 3
of 12 and 16: 2, 4
of 15 and 18: 3
#6a. List multiples of 2: 2, 4, 6, 8, 10, 12, 14, 16, 18, ⋯
b. List multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, ⋯
c. List multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, ⋯
#7.a. Multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, ⋯
3 $\nmid$ 4, 3 $\nmid$ 8. But 3|12. So 12 is a common multiple of 3 and 4.
b. Multiples of 5 are: 5, 10, 15, 20, 25, 30, ⋯
The first one of these divisible by 4 is 20. So 20 is a common multiple of 5 and 4.
c. Multiples of 4 were shown above. The first of these divisible by 6 is 12.
Therefore, 12 is a common multiple of 4 and 6.

2. Suppose a and b are whole numbers such that b is a multiple of a.
Then b = a·w for some whole number w.
GCF (a, b) = GCF(a, a·w) = a, since
a|a and a|aw.
No number larger than a divides a.
Therefore a must be the greatest common divisor of a and a·w. **QED■**

3. Assume p is a prime number.
GCF(a,p) must divide both a and p.
But since p is a prime, the only possible divisors are 1 and p.
If GCF(a,p) = 1, we are done.
If GCF(a,p) = p, then a is a multiple of p. **QED■**

4a.
$28 = 2^2 \cdot 7$
$63 = 3^2 \cdot 7$
GCF(28, 63) = 7

4b.
$104 = 2^3 \cdot 13$
$132 = 2^2 \cdot 3 \cdot 11$
GCF(104,132) = 2^2 = 4

4c.
$24 = 2^3 \cdot 3$
$56 = 2^2 \cdot 7$
$180 = 2^2 \cdot 3^2 \cdot 5$
GCF(24,56, 180) = 2^2 = 4

5. Use the Euclidean algorithm:

a.
91 = 1 · 52 + 39
52 = 1 · 39 + 13
39 = 3 · 13 + ☺↑
GCF(91, 52) = 13

b.
812 = 2 ·336 + 140
336 = 2 · 140 + 56
140 = 2 · 56 + 28
56 = 2 ·28 + ☺↑
GCF (812, 336) = 28

c.
2389485 = 40 ·59675 + 2485
59675 = 24· 2485 + 35
2485 = 71 · 35 + ☺↑
GCF(2389485,59675) = 35

6a.
$32 = 2^5$
$1024 = 2^{10}$
$\text{LCM}(32, 1024) = 2^{10}$
$\text{LCM}(32, 1024) = \text{LCM}(2^5, 2^{10}) = 2^{10}$

6b.
$24 = 2^3 \cdot 3$
$120 = 2^3 \cdot 3 \cdot 5$
$1056 = 2^5 \cdot 3 \cdot 11$
$\text{LCM}(24, 120, 1056) = 2^5 \cdot 3 \cdot 5 \cdot 11 = 5280$

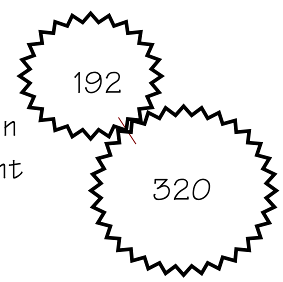
7a. On pages 38-9 of PT 5A, common multiples are used to find "common denominators" for adding and subtracting fractions.

7b.
$$\frac{2}{84} + \frac{5}{147} = \frac{1}{42} + \frac{5}{147} = \frac{1\cdot 7}{42\cdot 7} + \frac{5\cdot 2}{147\cdot 2} = \frac{7 + 10}{294} = \frac{17}{294}$$

I reduced the first fraction, because the numbers are smaller and I have less work.

8a.
$2n + 3 = \underline{\ 2\ } \cdot (n+1) + \underline{1}$
$n + 1 = \underline{\ n\ } \cdot (1) + 1$
$1 = \underline{\ 1\ } \cdot (1) + 0$ so $\text{GCF}(2n+3, n+1) = 1$

9. In order for the marks to be aligned, each gear must make a complete revolution, or complete multiple revolutions (back to the alignment position). Each revolution of the first gear requires the passage of 192 teeth; 320 teeth go by the alignment position for each revolution of the second gear.



If the marks are aligned now, then the first time they will again be aligned is after the passage of LCM(192, 320) teeth.

$192 = 2^5 \cdot 3$
$320 = 2^6 \cdot 5$
$\text{LCM} = 2^6 \cdot 3 \cdot 5$,

$2^6 \cdot 3 \cdot 5 = 192 \cdot 2 \cdot 5$
$2^6 \cdot 3 \cdot 5 = 320 \cdot 3$

which will require $2\cdot 5$ revolutions of the 192-toothed gear, and 3 revolutions of the 320-toothed gear.

10. The gym club is having an event, want all participants grouped neatly in rows (no "stragglers"). Whether in rows of 2, or 3, or 4, or5, or 6, or 7, or 8, there is always one left over! There are fewer than 1000 gymnasts (and, we assume, MORE THAN ONE). How many?

If there were NONE left over each time, then the smallest possible number (above 0) would be the
$\text{LCM}(2,3,4,5,6,7,8) = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840$

$2 = 2$
$3 = 3$
$4 = 2^2$
$5 = 5$
$6 = 2 \cdot 3$
$7 = 7$
$8 = 2^3$

Since the remainder is 1, rather than 0, the number is either 1, or 840+1 = 841.
If we assume the gym club has more than 1 gymnast, the least possible number of members is 841.