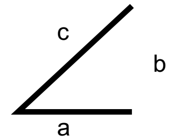
The Pythagoreans[1] believed that "Number rules the universe"; *everything* can be described in terms of numbers, and all numbers are rational– integers, or ratios of integers. Pythagoras proved[2] what is called the **Pythagorean Theorem**[3]:



$a^2 = b^2 + c^2$ **the sum of the squares of the sides of a right triangle is the square of the hypotenuse.**

Pythagoras' view of the describability of the universe in terms of rational numbers was contradicted-*destroyed*-by the very theorem which now bears his name. For if we construct a right triangle with sides of equal length 1, then we can demonstrate that the hypotenuse (*whose length ought to correspond to* ***some*** *number*), is $\sqrt{2}$, and $\sqrt{2}$ **cannot be expressed as a rational number**—cannot be expressed as the ratio of two integers. As we argue below:

First we note that every whole number has a unique prime factorization.

$$w = p_1^{r_1} p_2^{r_2} p_3^{r_3} p_4^{r_4} \cdots p_k^{r_k}$$

Then w^2 must have a prime factorization with the same primes to even powers:

$$w^2 = p_1^{2r_1} p_2^{2r_2} p_3^{2r_3} p_4^{2r_4} \cdots p_k^{2r_k}$$

Suppose

$$\sqrt{2} = \frac{p}{q}$$

where $p,q \in Z$, $q \neq 0$ [after all, that's what it would mean for $\sqrt{2}$ to be rational].

... and we may assume p/q *has been reduced to lowest terms*
(so p and q have NO common factors).

Multiplying both sides by q and squaring gives us

$$2q^2 = p^2 .$$

Which means 2 must be a factor of p^2, which in turn implies 2 must be a factor of p.

So p^2 has an odd power of 2 in its prime factorization... This is not possible.

Since every part of our argument is true,
the only flaw must be in the assumption we made in the first place–

that $\sqrt{2}$ can be written in rational form.

Pythagoras and his followers were devastated. Even today, we are troubled to discover there are numbers which cannot be written in the friendly form of a ratio of integers—but the irritation is diminished by the comforting thought that although such numbers exist, they are relatively few, and may be mostly ignored. NOT!

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1 students/disciples of the school of Pythagoras at Crotona in Southern Italy

2 ... or so he is credited. Pythagoras may have proved it, or perhaps one or some of his student followers.

3 though it was known to the Babylonians & Egyptians at least centuries earlier

Let's make a small digression. We now know these irrationals exist, and $\sqrt{2}$ is one of them, and can *not* be written as a ratio of integers; but what *is* the nature of an irrational? What is the decimal representation of an irrational number? We have previously seen that a *rational* number can be expressed in decimal form using the division algorithm, and that the decimal form either terminates or repeats. *Furthermore, any terminating or repeating decimal can be expressed as a ratio of two integers.*

Therefore:

The set of rational numbers is exactly the set of terminating and repeating decimals.
The Irrationals (*reals not expressible as ratio of two integers*)
are non-terminating, non-repeating decimals.

Consider, e.g. 1.10 200 300 040 000 500 000 600 000 070 000 000 800 000 000 900 000 000 010 000 000 000 011 ...
... or 1.12 123 123 412 345 123 456 123 456 712 345 678 123 456 789 123 456 789 101 234 567 891 011 ...

The sum of two rationals is rational. What about the sum of two irrationals?
This can be a bit difficult to test, since we have no general arithmetic algorithms for irrationals.

What is $\sqrt{2} + \sqrt{3}$? *(Hint: square that!)*
Is the sum of two irrationals always irrational?

...How about $\sqrt{2} + \sqrt{8}$? $-3\sqrt{2} + \sqrt{18}$?

What is the sum of a rational number and an irrational? Consider, for example, $\sqrt{2} + ½$. Intuitively, we have a non-terminating non-repeating decimal plus .5, the total of which should be non-terminating non-repeating. We can easily *prove* $\sqrt{2} + ½$ is irrational: suppose $\sqrt{2} + ½ =$ *a/b ...*)

The argument above showing $\sqrt{2}$ is irrational may be used to show that the square root of any prime (or of any composite number whose prime factors are not all to an even power) is not rational. Furthermore, there are other irrational numbers (*pi* and *e* for instance) that "occur" in numerous circumstances. In addition, for every irrational, we can construct a whole family of irrationals by adding any rational. (Irrational + rational is ...) Thus we can easily see there are at least as many irrational numbers as rationals. It was not until the end of the nineteenth century that a means of tackling the cardinality of infinite sets was devised to settle this question: How does the size of the set of irrationals compare with the size of the set of rationals?

Georg Cantor[4] was primarily responsible for the development of theory of sets & classes, with particular emphasis on the infinite. He devised the means of comparing cardinality of sets by mapping, or establishing a 1-1 correspondence between two sets to show they are of the same cardinality. (Recall our demonstrations that the cardinality of N is equivalent to that of Z.) Although the rationals are *dense* in the number line, the cardinality of Q is the same as that of N! But the cardinality of the irrationals is greater!

Here is Cantor's ingeniously simple proof:

Any set whose cardinality is that of N can be *listed*. (The act of *listing* establishes a 1-1 correspondence with N.) Suppose a "list" of all irrationals is presented. We will show that the list does not—*CANNOT*—contain all the irrationals, by constructing an irrational number that is not in the list.

Suppose the first few numbers in the alleged list are:

$m.d_1d_2d_3d_4d_5\ldots$	*e.g.*	*10.**1**450368...*	.**3**
$n.e_1e_2e_3e_4e_5\ldots$		*4.2**9**07863...*	**7**
$o.f_1f_2f_3f_4f_5\ldots$		*.00**0**6721...*	**2**
$p.g_1g_2g_3g_4g_5\ldots$		*.033**2**737...*	**4**
			⋱

We select a digit different from d_1 as the first decimal digit of our number; select a digit different from e_2 as the second digit of our number; select a digit different from f_3 as the third digit, and so on. The number so constructed cannot be the first number in the list because it differs in the first decimal place; cannot be the second number in the list because it differs in the second decimal place from that number; and so on. So it is *not in the list at all!* Thus the list is not complete.

Therefore, it is **not possible to list** all the irrationals, and thus they cannot be put into 1-1 correspondence with the natural numbers. **The cardinality of the set of irrationals is greater than the cardinality of N.** *Thus although the rationals are dense in the number line (no interval gaps), there are other numbers (irrationals) in the number line, and they far outnumber the rationals!*

Properties of Arithmetic Operations on REAL (rational & irrational) numbers:
Addition and multiplication on the set of all real numbers have the following properties:
CLOSURE; COMMUTATIVITY; ASSOCIATIVITY; IDENTITY (0 for + ; 1 for x);
INVERSES for + : for any decimal a, -a is the additive inverse.
INVERSES for x: for each decimal b other than 0, there is another decimal number, 1/b, such that bx(1/b) = 1
Subtraction and Division have the closure property except for division by 0.

[4] *1845-1918 Cantor's theory of the infinite so rocked the scientific and, particularly, mathematical world of his time that some antagonists were able to block his advancement; he was viewed as subversive by some, unbalanced by others. The bitterness of his life, coupled with insecurity bred in Cantor, and perhaps some genetic predisposition, led to a number of breakdowns for which Cantor was hospitalized. In the early twentieth century his work began to be recognized as the profoundly real work of a genius; this recognition was too little and too late.*