1. $P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5$

a. The degree of polynomial P is *5* and P must have *5* zeros (roots).

b. The y-intercept of the graph of P is *(0,−5)*. The number of vertical asymptotes of P is *0* ☆.

Polynomials don't have Vertical Asymptotes!!

c. According to Descartes' Rule of Signs, P can have *1* positive real zeros.

There is only one sign change in P(x) .
According to Descartes' Rule of Signs, there must be exactly 1 positive real root.

d. Similarly, P can have *0 or 2 or 4* negative real zeros.

P might have 4 or 2 or 0 negative real roots because :

$$P(-x) = 2(-x)^5 + 3(-x)^4 + 10(-x)^3 + 14(-x)^2 - 5$$
$$= -2x^5 + 3x^4 - 10x^3 + 14x^2 - 5$$

... four sign changes → 4 or 2 or 0 positive roots for P(−x) → 4 or 2 or 0 negative roots for P(x).

2. List all theoretically possible rational roots of the polynomial: $P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5$

Since this polynomial has all coefficients in the set of integers,
any rational roots must be of the form p/q
where p is a factor of the constant term (so p = ± 5 or 1) and
q is a factor of the leading coefficient (so q = ± 2 or 1).

Thus candidates for rational roots of this polynomial are:

$\pm \frac{1 \text{ or } 5}{1 \text{ or } 2}$ Generates the list: ± 1, 5, $^1/_2$, $^5/_2$

(There are 8 in all.)

Have trouble keeping p&q straight? Just ask yourself: what is the zero/root of P(x) = 2x+3 ? −3/2...
3 in the numerator,
2 in the denominator!

3. Construct the smallest degree polynomial with: real coefficients, roots −1, 1 and 2i, with leading coefficient 3. You may leave the polynomial in factored form.

Real coefficients plus having root 2i requires that it also have root −2i.
If "r" is a root, then (x −r) must be a factor, so...

$P(x) = A (x - -1) (x - 1) (x-2i) (x+2i) = A (x^2 - 1) (x^2 + 4) = A(x^4 + 3x^2 - 4)$

The leading coefficient now is A, so A must be 3. $P(x) = 3 (x^4 + 3x^2 - 4) = 3x^4 + 9x^2 - 12$

4. The polynomial $P(x) = 3x^4 - 8x^3 - 9x^2 + 16x + 6$ might have a zero at x= 2 or at x = 3.
Use synthetic division to demonstrate that one of these IS, indeed, a zero, and the other is NOT.
Identify which of these is a zero, and which is not.

2	3	−8	−9	16	6
		6	−4	−26	−20
	3	−2	−13	−10	14

3	3	−8	−9	16	6
		9	3	−18	−6
	3	1	−6	−2	0

Telling us that P(2) = 14

(So 2 is NOT a zero of P .)

Showing us that P(3) = 0...

(So 3 is a zero of P .)

...and, furthermore, $P(x) = (x - 3)(3x^3 + x^2 - 6x - 2)$

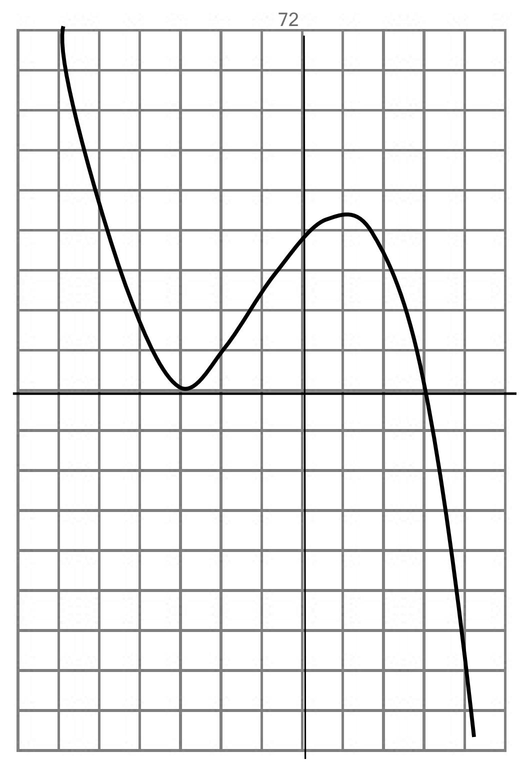
5. For each function below, list the equation(s) of the vertical and horizontal asymptote(s), if any. If there are none, write "none".

	Vertical asymptote(s)	Horizontal asymptote(s)
$g(x) = \dfrac{4x^2 - 1}{x^2 - 4}$	$x = -2$, $x = 2$	$y = 4$
$f(x) = \dfrac{3x + 9}{x^2 + 1}$	NONE	$y = 0$
$h(x) = \dfrac{6x^2 + 3}{2x + 1}$	$x = -½$	NONE Oblique: $y = 3x - {}^{3}/_{2}$

Vertical asymptotes of rational functions occur where the function grows unboundedly large because the denominator shrinks toward 0 while the numerator does not shrink to 0.
Horizontal asymptotes occur when the function settles toward a particular value as $x \to \infty$.
More on asymptotes- see next page.

6. The graph at right could be the graph of :

$P(x) = (x+3)^2 (3-x)$



$A(x) = (x-3)(x+3) + 23$ Parabola!

$B(x) = (x-3)(x+3) + 4$ Parabola!

$C(x) = (x-3)^2 (x+3)$ Roots 3,3,−3 and increasing

$D(x) = (x+3)^2 (3-x)$ Negative cubic, roots −3, −3

$E(x) = (x+3)^2 (x-3)^2$ Fourth degree!

$F(x) = -(x+3)^2 (x-3)^2$ Fourth degree!

This graph is NOT the graph of a polynomial of EVEN degree !!!
That leaves the two cubic polynomials as contenders.
But C has leading term $+x^3$ so would start low, run high.
Furthermore, C has a double zero at 3, and a single zero at −3.
D has leading term $-x^3$, AND has zeroes in the right places.

7. Find all the roots of the polynomial $P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5$
First we look to see if there are any advantageous factors. No, none this time.
Then we look for rational zeroes. Candidates are ± 1, 5, ${}^{1}/_{2}$, ${}^{5}/_{2}$ (Discussed in #2)

P(1) = 2 + 3 + 10 + 14 − 5 ...clearly not 0.

P(−1) = −2+3−10+14−5 = 0

−1	2	3	10	14	0	−5
	−2	−1	−9	−5	5	
−1	2	1	9	5	−5	0
		−2	1	−10	5	
½	2	−1	10	−5	0	
		1	0	5		
	2	0	10	0		

All this shows that $P(x) = (x+1)(x+1)(x - ½)(2x^2 + 10) = 2(x+1)(x+1)(x - ½)(x^2 + 5)$

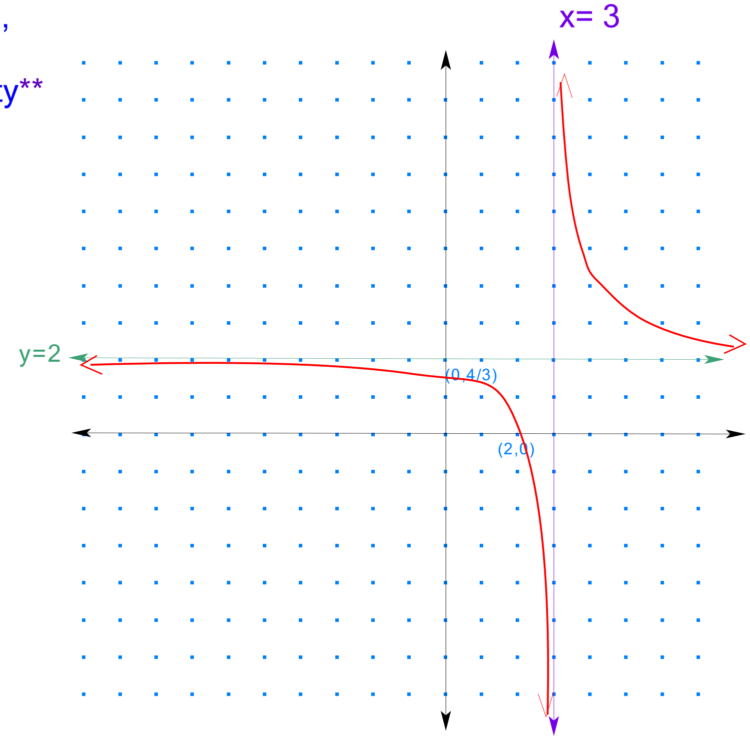
We obtain the last two zeroes from the quadratic factor $x^2 + 5$.

The zeroes of P are −1, −1, ½, and $\sqrt{5}$ and $-\sqrt{5}$.

Not on your test, but it SHOULD have been!

Not. Sketch the graph of $y = \frac{4 - 2x}{3 - x}$ Label all the intercepts & asymptotes.

This is a RATIONAL function , therefore:
Domain: y is defined for all values except where denominator = 0: x = 3
As x approaches 3, 4 – 2x approaches –2,
while 3 – x approaches 0,
so the quotient approaches ± infinity**
Thus there is a vertical asymptote
at x = 3.**



Any horizontal asymptote?
When x is large, y gets close to 2.
When x is large negative, same thing....
(More about this at *** below.)
Thus the function has a horizontal asymptote at y = 2.

y–intercept? when x = 0, y = 4/3

x–intercept? when x = 2, y = 0
(For P/Q to be 0, P must be 0.)

** More on the Vertical Asymptote:

As x approaches 3 from below e.g. take x = 2.5 , 2.8 , 2.9 , 2.99 , 2.9999 . . . → 3
y grows unboundedly large, negative: $y = \frac{-1}{.5}, \frac{-1.6}{-.2}, \frac{-1.8}{-.1}, \frac{-1.98}{-.01}, \frac{-1.9998}{-.0001}$ → inf

Similarly, computing function values for x that are just above 3,
demonstrates that the function values are positive and grow unboundedly large
(& we sometimes say "approaches infinity") as x-values approach 3 from above.

*** The right way to discover the horizontal (and any other non-vertical) asymptote:

...divide & conquer (the safe way to decipher these things):

$\frac{-2x + 4}{-x + 3} = \frac{2x - 4}{x - 3}$ (Now you divide! ...) $= 2 + \frac{2}{x - 3}$ As x → infinity , this fraction shrinks toward 0

& so f(x) approaches 2 + 0 (as x → infinity)

or, for a quick and only slightly dangerous view, observe that
when x is very large, 2x is so much larger than 4, and x is so much larger than 3, we can say the quotient behaves like $\frac{2x}{x}$ — which is just 2.

Vertical asymptotes of rational functions occur where the function grows unboundedly large (or)
because the denominator shrinks toward 0 while the numerator does not shrink.
This can only happen where the denominator of the rational function is 0.

Regarding #5:

	Vertical asymptote(s)	Horizontal asymptote(s)
$g(x) = \frac{4x^2 - 1}{x^2 - 4}$	$x = -2, x = 2$	$y = 4$
$f(x) = \frac{3x + 9}{x^2 + 1}$	NONE	$y = 0$
$h(x) = \frac{6x^2 + 3}{2x + 1}$	$x = -½$	NONE Oblique: $y = 3x - 3/2$

Vertical asymptotes of rational functions occur where the function grows unboundedly large

For g, the denominator, $x^2 - 4 = (x+2)(x-2)$, is 0 at −2 and 2. The numerator is not 0 at x = −2 or 2. Therefore, it is clear that f grows unboundedly large as x approaches −2 and as x approaches 2.

For f, the denominator, $x^2 + 1$, is never 0, so g cannot have a vertical asymptote.

For h, the denominator is 2x + 1. This is 0 when x = −½ . $6x^2 + 3$ does not shrink toward 0 as x approaches −½ . Thus there is a vertical asymptote at x = −½ .

Horizontal asymptotes of rational functions occur when the function values approach one particular number as x approaches infinity. This cannot occur if the degree of the numerator exceeds the degree of the denominator. (If degree of numerator = degree of denominator + 1, there is a linear asymptote that is neither horizontal nor vertical... called an oblique asymptote.)

g(x) approaches 4 as x approaches infinity... so g has HA y = 4.

Note that division demonstrates this: $g(x) = 4 + \frac{15}{x^2 - 4}$ and this fraction shrinks toward 0 as $x \to \infty$

f(x) approaches 0 as x approaches infinity... so f has HA y = 0.
This is clear because the degree of the denominator > degree of numerator, so these fractions shrink as the denominator grows faster than the numerator.

Degree of numerator exceeds degree of denominator of h, so h has no horizontal asymptote (note that h resembles y = 3x when x is very large).
... and, further, division again shows us the non-horizontal asymptote:

$$\begin{array}{r|l} & 3x - \tfrac{3}{2} \\ \hline 2x+1 & 6x^2 + 3 \\ & \underline{6x^2 + 3x} \\ & \quad -3x + 3 \\ & \quad \underline{-3x - \tfrac{3}{2}} \\ & \qquad \tfrac{9}{2} \end{array}$$

... telling us that:

$$h(x) = \frac{6x^2 + 3}{2x + 1} = 3x + 4.5 - \frac{15}{2x + 1}$$

Since the last part shrinks toward 0 as $x \to \infty$
h(x) must become ever closer to y = 3x + 4.5.
Thus h is asymptotic to the line y = 3x + 4.5 .
(This asymptote is called “oblique”.)

8. $P(x) = 4x^5 + 20x^4 - x - 5$

In each underlined space below, place the LETTER of the best completion of the statement.

a. The degree of polynomial P is 5 . And P must have 5 zeros (roots).

b. The graph of P has y-intercept –5 and has 0 vertical asymptotes.

c. According to Descartes' Rule of Signs, P can have 1 positive real zeros.

d. According to Descartes' Rule of Signs, P can have 4or2or0 negative real zeros.

9. According to the rational zeros theorem, only certain rational numbers may be zeros of P. Make a COMPLETE LIST of the possible rational zeros predicted by that theorem for the function:

$P(x) = 2x^4 - x^3 - 4x^2 - x - 6$ (This question does NOT ask if any are actual roots.)

According to the rational zeroes theorem, any rational zeroes of P must be $\pm \dfrac{1,2,3,6}{1,2}$

... which generates the list: $\pm$ 1,2,3,6, $^1/_2$, $^3/_2$

10. Use synthetic division to demonstrate that 2 is a zero of the polynomial $P(x) = x^3 + x - 10$.
Then find the remaining zeroes of the polynomial.

2	1	0	1	–10
		2	4	10
	1	2	5	0 ← Remainder 0 tells us P(2) = 0.

So $P(x) = (x - 2)(x^2 + 2x + 5)$ We find the remaining zeroes for the quadratic factor.

$$x^2 + 2x + 5 = 0$$

iff $$x = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

Thus the three zeroes of P are 2 and –1+2i and –1 – 2i .