1. \( P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5 \)

a. The degree of polynomial \( P \) is 5 and \( P \) must have 5 zeros (roots).

b. The y-intercept of the graph of \( P \) is \((0, -5)\). The number of vertical asymptotes of \( P \) is 0.

c. According to Descartes' Rule of Signs, \( P \) can have 1 positive real zeros.

There is only one sign change in \( P(x) \).
According to Descartes' Rule of Signs, there must be exactly 1 positive real root.

d. Similarly, \( P \) can have 0 or 2 or 4 negative real zeros.

\( P \) might have 4 or 2 or 0 negative real roots because:

\[
\begin{align*}
P(-x) &= 2(-x)^5 + 3(-x)^4 + 10(-x)^3 + 14(-x)^2 - 5 \\
&= -2x^5 + 3x^4 - 10x^3 + 14x^2 - 5 \\
&= \ldots \text{four sign changes} \\
&\rightarrow 4 \text{ or } 2 \text{ or } 0 \text{ positive roots for } P(-x) \rightarrow 4 \text{ or } 2 \text{ or } 0 \text{ negative roots for } P(x).
\end{align*}
\]

2. List all theoretically possible rational roots of the polynomial: \( P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5 \)

Since this polynomial has all coefficients in the set of integers, any rational roots must be of the form \( \frac{p}{q} \) where \( p \) is a factor of the constant term (so \( p = \pm 5 \) or 1) and \( q \) is a factor of the leading coefficient (so \( q = \pm 2 \) or 1).

Thus candidates for rational roots of this polynomial are:

\[
\pm \frac{1}{1} \text{ or } \frac{5}{2}, \frac{1}{2}, \frac{5}{2}
\]

Generates the list: \( \pm 1, 5, \frac{1}{2}, \frac{5}{2} \)

(There are 8 in all.)

3. Construct the smallest degree polynomial with: real coefficients, roots \(-1, 1\) and \(2i\), with leading coefficient 3. You may leave the polynomial in factored form.

Real coefficients plus having root \(2i\) requires that it also have root \(-2i\).

If \(r\) is a root, then \((x - r)\) must be a factor, so...

\[
P(x) = A(x - 1)(x - 1)(x - 2i)(x + 2i) = A(x^2 - 1)(x^2 + 4) = A(x^4 + 3x^2 - 4)
\]

The leading coefficient now is \(A\), so \(A\) must be 3.

\[
P(x) = 3(x^4 + 3x^2 - 4) = 3x^4 + 9x^2 - 12
\]

4. The polynomial \( P(x) = 3x^4 - 8x^3 - 9x^2 + 16x + 6 \) might have a zero at \( x = 2 \) or at \( x = 3 \).

Use synthetic division to demonstrate that one of these IS, indeed, a zero, and the other is NOT. Identify which of these is a zero, and which is not.

\[
\begin{array}{c|ccccc}
2 & 3 & -8 & -9 & 16 & 6 \\
& & 6 & -4 & -26 & -20 \\
\hline
& 3 & -2 & -13 & -10 & 14
\end{array}
\]

Telling us that \( P(2) = 14 \)

(\text{So } 2 \text{ is NOT a zero of } P.)

\[
\begin{array}{c|ccccc}
3 & 3 & -8 & -9 & 16 & 6 \\
& & 9 & 3 & -18 & -6 \\
\hline
& 3 & 1 & -6 & -2 & 0
\end{array}
\]

Showing us that \( P(3) = 0 \ldots \)

(\text{So } 3 \text{ is a zero of } P.)

...and, furthermore, \( P(x) = (x - 3)(3x^3 + x^2 - 6x - 2) \)
5. For each function below, list the equation(s) of the vertical and horizontal asymptote(s), if any. If there are none, write “none”.

<table>
<thead>
<tr>
<th>Function</th>
<th>Vertical asymptote(s)</th>
<th>Horizontal asymptote(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x) = \frac{4x^2 - 1}{x^2 - 4}$</td>
<td>$x = -2, x = 2$</td>
<td>$y = 4$</td>
</tr>
<tr>
<td>$f(x) = \frac{3x + 9}{x^2 + 1}$</td>
<td>NONE</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$h(x) = \frac{6x^2 + 3}{2x + 1}$</td>
<td>$x = -\frac{1}{2}$</td>
<td>NONE, $y = 3x - \frac{3}{2}$</td>
</tr>
</tbody>
</table>

Vertical asymptotes of rational functions occur where the function grows unboundedly large because the denominator shrinks toward 0 while the numerator does not shrink to 0.
Horizontal asymptotes occur when the function settles toward a particular value as $x \to \infty$.
More on asymptotes—see next page.

6. The graph at right could be the graph of:

- $P(x) = (x + 3)^2 (3 - x)$
- $A(x) = (x - 3) (x+3) + 23$ Parabola!
- $B(x) = (x - 3) (x+3) + 4$ Parabola!
- $C(x) = (x - 3)^2 (x+3)$ Roots 3,3,−3 and increasing
- $D(x) = (x + 3)^2 (3 - x)$ Negative cubic, roots −3, −3
- $E(x) = (x + 3)^2 (x - 3)^2$ Fourth degree!
- $F(x) = - (x + 3)^2 (x - 3)^2$ Fourth degree!

This graph is NOT the graph of a polynomial of EVEN degree !!!
That leaves the two cubic polynomials as contenders.
But C has leading term $+x^3$ so would start low, run high.
Furthermore, C has a double zero at 3, and a single zero at $-3$.
D has leading term $-x^3$, AND has zeroes in the right places.

7. Find all the roots of the polynomial $P(x) = 2x^5 + 3x^4 + 10x^3 + 14x^2 - 5$

First we look to see if there are any advantageous factors. No, none this time.
Then we look for rational zeroes. Candidates are $\pm 1, 5, \frac{1}{2}, \frac{5}{2}$ (Discussed in #2)

$$P(1) = 2 + 3 + 10 + 14 - 5 \cdots \text{clearly not 0.}$$
$$P(-1) = -2 + 3 - 10 + 14 - 5 = 0$$

All this shows that $P(x) = (x+1) (x+1) (x - \frac{1}{2}) (2x^2 + 10) = 2 (x+1) (x+1) (x - \frac{1}{2}) (x^2 + 5)$
We obtain the last two zeroes from the quadratic factor $x^2 + 5$.
The zeroes of P are $-1, -1, \frac{1}{2}$, and $\sqrt{5}$ and $-\sqrt{5}$.
Sketch the graph of \( y = \frac{4 - 2x}{3 - x} \). Label all the intercepts & asymptotes.

This is a rational function, therefore:
Domain: \( y \) is defined for all values except where denominator = 0: \( x = 3 \)
As \( x \) approaches 3, \( 4 - 2x \) approaches -2,
while \( 3 - x \) approaches 0,
so the quotient approaches ± infinity**
Thus there is a vertical asymptote at \( x = 3 \).**

Any horizontal asymptote?
When \( x \) is large, \( y \) gets close to 2.
When \( x \) is large negative, same thing....
(More about this at *** below.)
Thus the function has a horizontal asymptote at \( y = 2 \).

\( y \)-intercept? when \( x = 0, y = \frac{4}{3} \)

\( x \)-intercept? when \( x = 2, y = 0 \)
(For \( P/Q \) to be 0, \( P \) must be 0.)

** More on the Vertical Asymptote:

As \( x \) approaches 3 from below e.g. take \( x = 2.5, 2.8, 2.9, 2.99, 2.9999 \ldots \rightarrow 3 \)
y grows unboundedly large, negative:
\( y = -1, -1.6, -1.8, -1.98, -1.998, -1.9998 \ldots \rightarrow \text{inf} \)

Similarly, computing function values for \( x \) that are just above 3,
demonstrates that the function values are positive and grow unboundedly large
(& we sometimes say “approaches infinity”) as \( x \)-values approach 3 from above.

*** The right way to discover the horizontal (and any other non-vertical) asymptote:

...divide & conquer (the safe way to decipher these things):
\[
\frac{-2x + 4}{-x + 3} = \frac{2x - 4}{x - 3} \quad \text{(Now you divide! ...)} = 2 + \frac{2}{x - 3}
\]
As \( x \rightarrow \text{infinity} \),
this fraction shrinks toward 0
& so \( f(x) \) approaches \( 2 + 0 \) (as \( x \rightarrow \text{infinity} \))
or, for a quick and only slightly dangerous view, observe that
when \( x \) is very large, \( 2x \) is so much larger than 4, and \( x \) is so much larger than 3, we can say the
quotient behaves like \( \frac{2x}{x} \) — which is just 2.

Vertical asymptotes of rational functions occur where the function grows unboundedly large (or)
because the denominator shrinks toward 0 while the numerator does not shrink.
This can only happen where the denominator of the rational function is 0.
Regarding #5:

<table>
<thead>
<tr>
<th>g(x) = \frac{4x^2 - 1}{x^2 - 4}</th>
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<td>(y = 4)</td>
<td></td>
</tr>
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<table>
<thead>
<tr>
<th>f(x) = \frac{3x + 9}{x^2 + 1}</th>
<th>Vertical asymptote(s)</th>
<th>Horizontal asymptote(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NONE</td>
<td>(y = 0)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>h(x) = \frac{6x^2 + 3}{2x + 1}</th>
<th>Vertical asymptote(s)</th>
<th>Horizontal asymptote(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = -\frac{1}{2})</td>
<td>Oblique: (y = 3x - \frac{3}{2})</td>
<td></td>
</tr>
</tbody>
</table>

Vertical asymptotes of rational functions occur where the function grows unboundedly large.

For \(g\), the denominator, \(x^2 - 4 = (x+2)(x-2)\), is 0 at -2 and 2. The numerator is not 0 at \(x = -2\) or 2. Therefore, it is clear that \(g\) grows unboundedly large as \(x\) approaches -2 and as \(x\) approaches 2.

For \(f\), the denominator, \(x^2 + 1\), is never 0, so \(g\) cannot have a vertical asymptote.

For \(h\), the denominator is \(2x + 1\). This is 0 when \(x = -\frac{1}{2}\). \(6x^2 + 3\) does not shrink toward 0 as \(x\) approaches \(-\frac{1}{2}\). Thus there is a vertical asymptote at \(x = -\frac{1}{2}\).

Horizontal asymptotes of rational functions occur when the function values approach one particular number as \(x\) approaches infinity. This cannot occur if the degree of the numerator exceeds the degree of the denominator. (If degree of numerator = degree of denominator + 1, there is a linear asymptote that is neither horizontal nor vertical... called an oblique asymptote.)

\(g(x)\) approaches 4 as \(x\) approaches infinity... so \(g\) has HA \(y = 4\).

Note that division demonstrates this: 
\[
g(x) = 4 + \frac{15}{x^2 - 4}
\]
and this fraction shrinks toward 0 as \(x \to \infty\)

\(f(x)\) approaches 0 as \(x\) approaches infinity... so \(f\) has HA \(y = 0\).
This is clear because the degree of the denominator > degree of numerator, so these fractions shrink as the denominator grows faster than the numerator.

Degree of numerator exceeds degree of denominator of \(h\), so \(h\) has no horizontal asymptote (note that \(h\) resembles \(y = 3x\) when \(x\) is very large).

... and, further, division again shows us the non-horizontal asymptote:

\[
\begin{align*}
2x+1 & \frac{3x}{6x^2 + 3} + \frac{3}{6x^2 + 3} \\
& \frac{-3x + 3}{6x^2 + 3} - \frac{3x}{\frac{3}{2}} \\
& \frac{-3x}{\frac{3}{2}} \\
& \frac{9/2}{9/2}
\end{align*}
\]

... telling us that:
\[
h(x) = \frac{6x^2 + 3}{2x + 1} = 3x + 4.5 - \frac{15}{2x + 1}
\]

Since the last part shrinks toward 0 as \(x \to \infty\) \(h(x)\) must become ever closer to \(y = 3x + 4.5\). Thus \(h\) is asymptotic to the line \(y = 3x + 4.5\). (This asymptote is called “oblique”.)
8. \( P(x) = 4x^5 + 20x^4 - x - 5 \)

In each underlined space below, place the LETTER of the best completion of the statement.

a. The degree of polynomial \( P \) is \( \boxed{5} \). And \( P \) must have \( \boxed{5} \) zeros (roots).

b. The graph of \( P \) has y-intercept \( \boxed{-5} \) and has \( \boxed{0} \) vertical asymptotes.

c. According to Descartes’ Rule of Signs, \( P \) can have \( \boxed{1} \) positive real zeros.

d. According to Descartes’ Rule of Signs, \( P \) can have \( \boxed{4\text{or2or0}} \) negative real zeros.

9. According to the rational zeros theorem, only certain rational numbers may be zeros of \( P \). Make a COMPLETE LIST of the possible rational zeros predicted by that theorem for the function:

\[ P(x) = 2x^3 - x^2 - 4x - 6 \]

(This question does NOT ask if any are actual roots.)

According to the rational zeroes theorem, any rational zeroes of \( P \) must be \( \pm \frac{1,2,3,6}{1,2} \)

... which generates the list: \( \pm 1,2,3,6, \frac{1}{2}, \frac{3}{2} \)

10. Use synthetic division to demonstrate that 2 is a zero of the polynomial \( P(x) = x^3 + x - 10 \). Then find the remaining zeroes of the polynomial.

\[
\begin{array}{c|cccc}
2 & 1 & 0 & 1 & -10 \\
 & & 2 & 4 & 10 \\
\hline
1 & 2 & 5 & 0 & \text{ Remainder 0 tells us } P(2) = 0.
\end{array}
\]

So \( P(x) = (x - 2) (x^2 + 2x + 5) \) We find the remaining zeroes for the quadratic factor.

\[ x^2 + 2x + 5 = 0 \]

iff \( x = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i \)

Thus the three zeroes of \( P \) are 2 and \(-1+2i\) and \(-1 - 2i\).