

1. Graph a polynomial function. Label all intercepts and describe the end behavior.

a. $P(x) = x^4 - 2x^3 - 15x^2$.

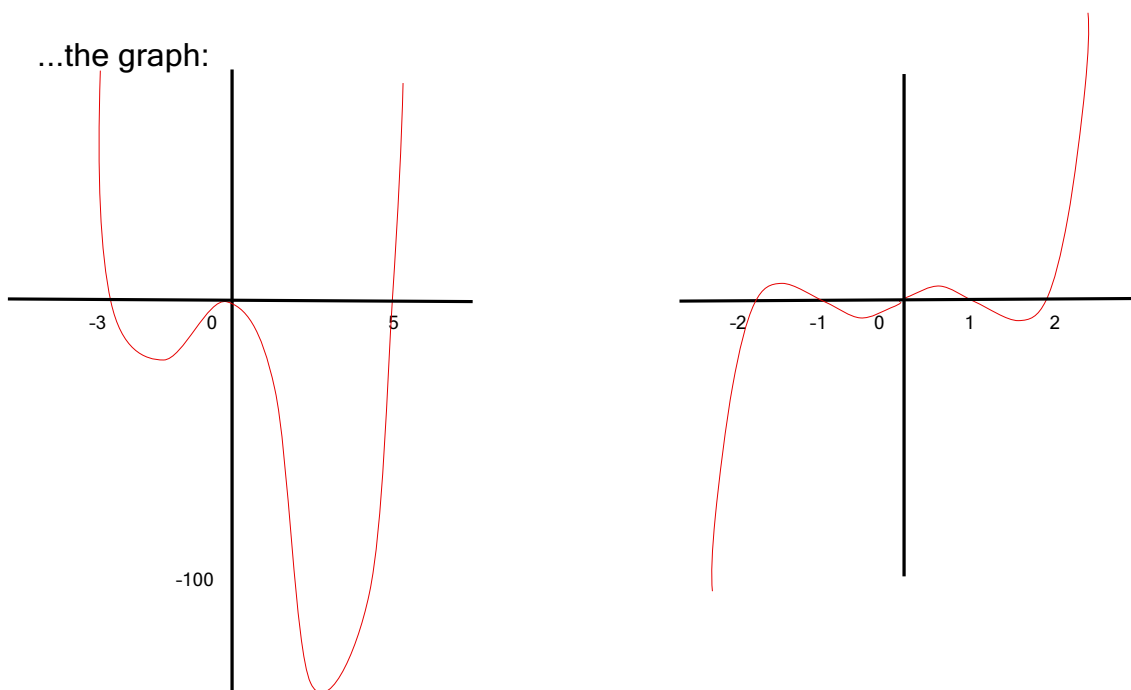
(1) Domain = \mathbb{R} , of course (since this is a polynomial function... domain must be \mathbb{R} .)

(2) y-intercept: $P(0) = 0^4 - 2 \cdot 0^3 - 15 \cdot 0^2 = 0$

(3) x-intercepts: $x^4 - 2x^3 - 15x^2 = 0$
 $x^2 (x^2 - 2x - 15) = 0$
 $x^2 (x - 5)(x + 3) = 0$
 $x = 0$ or 0 or 5 or -3

(4) the outer limits: As $x \rightarrow \infty$, x^4 dominates the other terms, so $f(x) \rightarrow \infty$.
 (i.e. $f(x)$ grows unboundedly large)
 Likewise, as $x \rightarrow -\infty$, x^4 , dominating the other terms, takes $f(x) \rightarrow \infty$

(5) ...the graph:



b. $P(x) = x^5 - 5x^3 + 4x$.

(1) Domain = \mathbb{R}

(2) y-intercept: $P(0) = 0$

(3) x-intercepts : $x^5 - 5x^3 + 4x = 0$
 when $x(x^4 - 5x^2 + 4) = 0$
 $x(x^2 - 4)(x^2 - 1) = 0$
 $x(x+2)(x-2)(x+1)(x-1) = 0$
 $x = 0, -2, 2, -1, 1$

(4) outer limits: as $x \rightarrow \infty$, $f(x)$, dominated by x^5 , $\rightarrow \infty$
 as $x \rightarrow -\infty$, $f(x)$, again ruled by x^5 , $\rightarrow -\infty$

(5) see above, right. P is an odd function.

Note the curve is smoother than this pathetic illustration indicates.
 It is just difficult to draw these curves with a mouse and flaky software.
 After the seventeenth try, patience vanishes.

2. Use polynomial long division.

EG: Divide ... $\frac{6x^4 + 2x^2 + 22x}{2x^2 + 5}$

$$\begin{array}{r}
 3x^2 \quad - \frac{13}{2} \\
 2x^2 + 5 \overline{) 6x^4 + 0x^3 + 2x^2 + 22x + 0} \\
 \underline{6x^4 + 15x^2} \\
 -13x^2 \\
 \underline{-13x^2 } \\
 22x + \frac{65}{2}
 \end{array}$$

SO $\frac{6x^4 + 2x^2 + 22x}{2x^2 + 5} = 3x^2 - \frac{13}{2} + \frac{22x + \frac{65}{2}}{2x^2 + 5}$

OR $6x^4 + 2x^2 + 22x = (3x^2 - \frac{13}{2})(2x^2 + 5) + 22x + \frac{65}{2}$

Notice the last statement shows the way to check your division results!!

3. a. The graph shows us there is a repeated zero*, of even order, at $x = 0$, and a simple zero at $x = 3$. It also tells us $P(2)$ must be 8.

The zeroes inform us that $(x-0)(x-0)(x-3)$ must be part of the factorization of P . Therefore, $P(x) = x \cdot x \cdot (x-3) \cdot (\text{other factors})$. The degree of P (3) tells us that the "other factors" can be only constant..

Stated briefly, we know:

Roots are 0, 0, and 3 & polynomial has degree 3, so

$$P(x) = K x^2 (x-3)$$

Since $P(2) = 8$:

$$8 = P(2) = K \cdot 2^2(2-3) = -4K, \text{ so } K \text{ must be } -2.$$

$$\text{Therefore, } P(x) = -2 x^2 (x-3), \text{ or } P(x) = -2x^3 + 6x^2$$

- b. Find a second-degree polynomial with real coefficients with roots -3 and 2, passing through (4,3).

r is a root of a polynomial P if, and only if, $(x - r)$ is a factor of $P(x)$.

Having roots -3 and 2 requires that both $(x - -3)$ and $(x - 2)$ be factors.

Since P must be a 2nd-degree polynomial, the only other possible factor is a constant. So:

$$P(x) = A(x + 3)(x - 2) \quad \text{where } A \text{ is some currently undetermined constant.}$$

The additional requirement that the curve contain (4,3) determines A :

$$P(4) = A(4+3)(4-2) = A(7)(2) = 14A$$

But $P(4)$ must be 3... so $14A$ must be 3. therefore $A = 3/14$ and

$$P(x) = (3/14)(x + 3)(x - 2) = (3/14)(x^2 + x - 6)$$

- c. Find a fourth degree polynomial with real coefficients, with roots $2i$ and $1 - i$, and constant term 16.

The complex roots of a polynomial with real coefficients must occur in conjugate pairs.

Since we are required to find a polynomial with real coefficients, and since $1 - i$ is a root, its conjugate, $1 + i$ must also be a root.

Likewise, since $2i$ is a root, $-2i$ must also be a root. Therefore:

$$\begin{aligned} P(x) &= A(x - (1 + i))(x - (1 - i))(x - 2i)(x + 2i) \\ &= A(x^2 - (1 + i + 1 - i)x + (1 + i)(1 - i))(x^2 - 4i^2) \\ &= A(x^2 - 2x + 2)(x^2 + 4) \end{aligned}$$

The additional requirement that the constant term be 16 determines A.

$$\begin{aligned} A(2)(4) &\text{ must be 16.... thus A must be 2, and } P(x) = 2(x^2 - 2x + 2)(x^2 + 4) \\ &= 2(x^4 - 2x^3 + 6x^2 - 8x + 8) \\ &= 2x^4 - 4x^3 + 12x^2 - 16x + 16 \end{aligned}$$

4. Find all the zeros of a polynomial function.

- a. For $P(x) = 5x^3 - 22x^2 + 18x - 4$, list all the theoretically possible rational roots of P; and use synthetic division to locate one; then find the remaining roots.

RATIONAL roots of a polynomial with INTEGER coefficients must be of the form p/q where p divides the constant term and q divides the leading coefficient.

So any rational roots of $5x^3 - 22x^2 + 18x - 4$ must be of the form:

$$\frac{\pm 4 \text{ or } \pm 2 \text{ or } \pm 1}{\pm 5 \text{ or } \pm 1} \quad \text{which yields: } \pm 1 \pm 2 \pm 4 \pm \frac{1}{5} \pm \frac{2}{5} \pm \frac{4}{5}$$

TESTING we divide $5x^3 - 22x^2 + 18x - 4$ by $x - r$ for each potential root r:

$$\begin{array}{r|rrrr} 1 & 5 & -22 & 18 & -4 \\ & & 5 & -17 & 1 \\ \hline & 5 & -17 & 1 & -3 \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} -1 & 5 & -22 & 18 & -4 \\ & & -5 & 27 & \\ \hline & 5 & -27 & & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} 2 & 5 & -22 & 18 & -4 \\ & & 10 & -24 & -12 \\ \hline & 5 & -12 & -6 & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} -2 & 5 & -22 & 18 & -4 \\ & & -10 & -17 & 1 \\ \hline & 5 & -32 & 1 & -3 \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} 4 & 5 & -22 & 18 & -4 \\ & & 20 & -8 & 40 \\ \hline & 5 & -2 & 10 & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} -4 & 5 & -22 & 18 & -4 \\ & & -10 & 128 & -12 \\ \hline & 5 & -32 & & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} 1/5 & 5 & -22 & 18 & -4 \\ & & 1 & -21/5 & 69/25 \\ \hline & 5 & -21 & 69/5 & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} -1/5 & 5 & -22 & 18 & -4 \\ & & -1 & -23/5 & \\ \hline & 5 & -23 & 67/5 & \end{array} \quad \text{⊗}$$

$$\begin{array}{r|rrrr} 2/5 & 5 & -22 & 18 & -4 \\ & & 2 & -8 & 4 \\ \hline & 5 & -20 & 10 & 0 \end{array}$$



$$\text{So } P(x) = (x - 2/5)(5x^2 - 20x + 10) = (5x - 2)(x^2 - 4x + 2)$$

$$\text{For the remaining roots we solve: } x^2 - 4x + 2 = 0$$

$$\dots \text{ using the quadratic formula: } x = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$$

PS $P(-x) = -5x^3 - 22x^2 - 18x - 4$, so, by Descartes' Rule of Signs, there are no negative real roots.

- b. Find all the roots of $P(x) = 4x^5 + 15x^3 - 4x$

$$4x^5 + 15x^3 - 4x =$$

$$x(4x^4 + 15x^2 - 4) =$$

$$x(4x^2 - 1)(x^2 + 4) =$$

$$x(2x+1)(2x-1)(x+2i)(x-2i)$$

So the roots of P are: $0, -\frac{1}{2}, \frac{1}{2}, -2i, 2i$

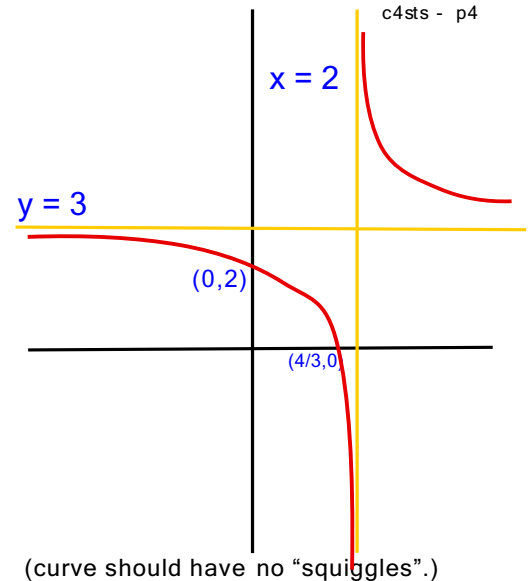
5. a. $f(x) = \frac{3x - 4}{x - 2}$

- Find the x and y intercepts.
 $f(0) = 2$; $f(x) = 0$ when $3x - 4 = 0 \dots x = \frac{4}{3}$
- Find the equation of the vertical asymptote.
 Near $x = 2$, numerator $\rightarrow 2$, denominator $\rightarrow 0$,
 so fraction $\rightarrow \pm\infty \dots$ vertical asymptote at $x = 2$.
- Find the equation of the horizontal asymptote.
 As $x \rightarrow \infty$, $(3x - 4)/(x - 2) \rightarrow 3$
 so $y = 3$ is the horizontal asymptote.

- Division also shows us:

$$\frac{3x - 4}{x - 2} = 3 + \frac{2}{x - 2}, \text{ which would have to be ...}$$

$1/x$, shifted 2 units right, stretched vertically by factor 2,
 then shifted upwards 3 units.



b. $f(x) = \frac{x^2 - 3x - 4}{x - 2} = \frac{(x - 4)(x + 1)}{x - 2}$

- Intercepts: $f(0) = 2$.
 $f(x) = 0$ when $x = 4, -1$.
- Vertical asymptote
 At $x = 2$, for the usual reason.

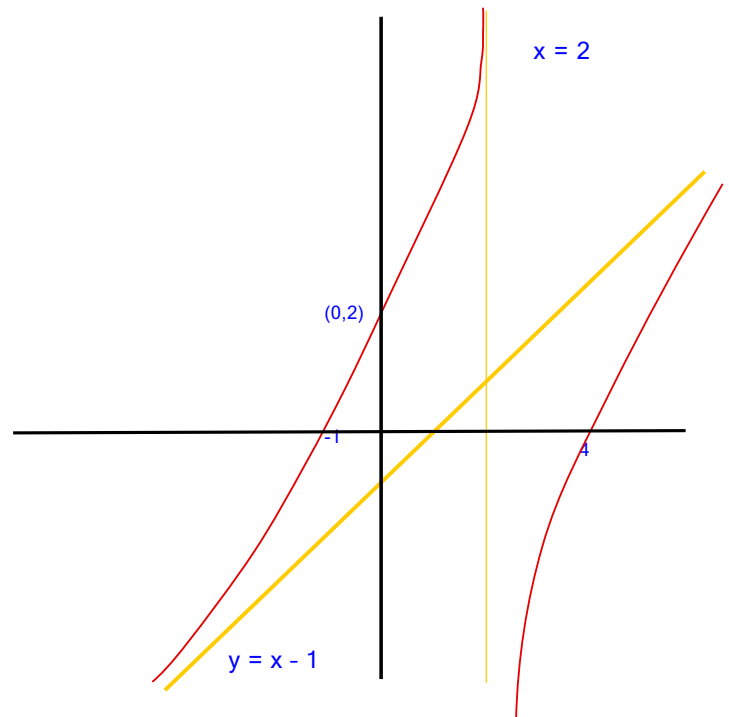
- Horizontal asymptote

None.

Divide to see other asymptote:

$$\frac{x^2 - 3x - 4}{x - 2} = x - 1 + \frac{-6}{x - 2}$$

So we can see that $y = x - 1$ is an oblique linear asymptote for $y = f(x)$.

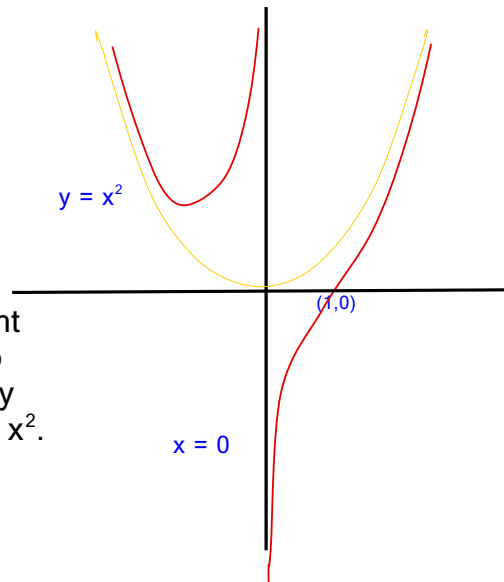


$$c. \quad f(x) = \frac{x^3 - 1}{x} = \frac{(x-1)(x^2 + x + 1)}{x}$$

x-intercept (1,0)**

vertical asymptote at $x = 0$

$$f(x) = \frac{x^3 - 1}{x} = x^2 - \frac{1}{x}$$



... this shows the given f is $x^2 +$ an amount that is very small when x is very large. So when x is large, the graph of $y = f(x)$ is very close to the graph of the familiar curve $y = x^2$. And as $x \rightarrow \infty$, the difference \rightarrow zero. This is asymptotic behavior.

**of course, we determined that the remaining roots are complex:

$$x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

6. The Remainder Theorem gives us the answers:

If polynomial $P(x) = (x - c) Q(x) + R(x)$ where Q & R are the quotient and remainder polynomials for $P \div (x - c)$, then $R(x)$ is a constant, and $P(c) =$ that constant.

a. Find the value of $P(x) = 2x^5 - 20x^4 - 20x^3 - 20x^2 - 20x - 22$ without raising 11 to a power.

$$\begin{array}{r|rrrrrr} 11 & 2 & -20 & -20 & -20 & -20 & -22 \\ & & 22 & 22 & 22 & 22 & \\ \hline & 2 & 2 & 2 & 2 & 2 & 0 \end{array}$$

This tells us that $P(11) = 0$, and 11 is a root, and $(x - 11)$ is a factor of P .
In fact, $P(x) = (x - 11)^2 (x^4 + x^3 + x^2 + x + 1)$

b. If $P(x) = (x - 2)(x^3 - 4x^2 + 7x + 13) + 7$, what is $P(2)$?

$$P(2) = (2 - 2) (\text{whatever}) + 7$$

$$= 0 + 7 \quad \dots \text{just as the remainder theorem says.}$$