1. \( P(x) = 2x^5 - 2x^4 - 3x^3 - 6x^2 - 2x + 4 \)

a. The degree of polynomial \( P \) is \( 5 \) and \( P \) must have \( 5 \) zeros (roots).

b. The y-intercept of the graph of \( P \) is \((0,4)\). The number of vertical asymptotes of \( P \) is \( 0 \).

c. According to Descartes’ Rule of Signs, \( P \) can have \( 0 \) or \( 2 \) positive real zeros.

   There are two sign changes in \( P(x) = 2x^5 - 2x^4 - 3x^3 - 6x^2 - 2x + 4 \).
   According to Descartes’ Rule of Signs, there can be either 2 or 0 positive roots.

d. Similarly, \( P \) can have \( 1 \) or \( 3 \) negative real zeros.

   \( P \) cannot have 3 or 1 negative real roots because:
   \[
   P(-x) = 2(-x)^5 - 2(-x)^4 - 3(-x)^3 - 6(-x)^2 - 2(-x) + 4 \\
   = -2x^5 - 2x^4 + 3x^3 - 6x^2 + 2x + 4 \\
   
   \text{... three sign changes, so 3 or 1 positive roots for } P(-x), \text{ so 3 or 1 negative roots for } P(x).
   \]

2. List all theoretically possible rational roots of the polynomial: \( P(x) = 4x^4 + 4x^3 + 9x^2 + 12x - 9 \)

   Since this polynomial has all coefficients in the set of integers, any rational roots must be of the form \( \frac{p}{q} \) where \( p \) is a factor of the constant term (so \( p = \pm 9, 3 \) or \( 1 \)) and \( q \) is a factor of the leading coefficient (so \( q = \pm 4, 2 \), or \( 1 \)).

   Thus candidates for rational roots of this polynomial are:
   \[
   \pm 9 \quad \pm 3 \quad \pm 1 \quad \pm 9/2 \quad \pm 3/2 \quad \pm 1/2 \quad \pm 9/4 \quad \pm 3/4 \quad \pm 1/4
   \]
   (There are 18 in all.)

3. Construct the smallest polynomial with: real coefficients, roots \( -2 \) and \( 2 \) and \( 3 + i \), with leading coefficient 5. You may leave the polynomial in factored form.

   Real coefficients + having root \( 3+i \) requires that it also have root \( 3-i \).
   If “\( r \)” is a root, then \( (x - r) \) must be a factor, so...
   \[
   P(x) = A \left((x - 2)(x - 2)(x - (3+i))(x - (3-i))\right)
   \]

4. The polynomial \( P(x) = 2x^4 - 3x^3 - 7x^2 - 8x + 6 \) might have a zero at \( x = \frac{1}{2} \) or at \( x = 2 \).
   Use synthetic division to demonstrate that one of these IS, indeed, a zero, and the other is NOT. Identify which of these is a zero, and which is not.

   Telling us that \( 2x^4 - 3x^3 - 7x^2 - 8x + 6 \) = \( (x - \frac{1}{2})(2x^3 - 2x^2 - 8x - 12) \)
   (So \( \frac{1}{2} \) is a zero of \( P \).)
   Showing us that \( P(2) = -3 \)
   (So \( 2 \) is NOT a zero of \( P \).)
5. For each function below, list the equation(s) of the vertical and horizontal asymptote(s), if any. If there are none, write “none”.

<table>
<thead>
<tr>
<th>Function</th>
<th>Vertical asymptote(s)</th>
<th>Horizontal asymptote(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) = \frac{x^2 - 9}{x^2 + 4} )</td>
<td>NONE</td>
<td>( y = 1 )</td>
</tr>
<tr>
<td>( f(x) = \frac{3x^4 + x}{x^2 - 9} )</td>
<td>( x = 3 ) and ( x = -3 )</td>
<td>NONE</td>
</tr>
<tr>
<td>( h(x) = \frac{6x + 3}{x^2 + 2x + 1} )</td>
<td>( x = -1 )</td>
<td>( y = 0 )</td>
</tr>
</tbody>
</table>

Vertical asymptotes of rational functions occur where the function grows unboundedly large (or 0 or -
because the denominator shrinks toward 0 while the numerator does not shrink. This can only happen where the denominator of the rational function is 0.

For \( g \), the denominator, \( x^2 + 4 \), is never 0, so \( g \) cannot have a vertical asymptote.

For \( f \), the denominator, \( x^2 - 9 = (x+3)(x-3) \), is 0 at -3 and 3. The numerator is not 0 at \( x = -3 \) or 3. Therefore, it is clear that \( f \) grows unboundedly large as \( x \) approaches -3 and as \( x \) approaches 3.

For \( h \), the denominator is \( x^2 + 2x + 1 = (x+1)^2 \). This is 0 when \( x = -1 \). 6x+3 does not shrink toward 0 as \( x \) approaches -1. Thus there is a vertical asymptote at \( x = -1 \).

Horizontal asymptotes of rational functions occur when the function values approach one particular number as \( x \) approaches infinity. This cannot occur when the degree of the numerator exceeds the degree of the denominator.

So \( f \) has no horizontal asymptote (\( f \) resembles \( y = 3x^2 \) when \( x \) is very large).

\( g(x) \) approaches 1 as \( x \to \infty \) so \( g \) has HA \( y = 1 \).

\( h(x) \) approaches 0 as \( x \to \infty \) so \( h \) has HA \( y = 0 \).

6. Sketch the graph of \( y = P(x) = \frac{1}{2} \ (x + 2)^2 \ (x - 3) \)

Polynomial, domain is all of \( R \), continuous, etc.

\( y \)-intercept: \( P(0) = \frac{1}{2} \ (0+2)^2 \ (0 - 3) = -6 \)

\( x \)-intercept(s): \( P(x) = 0 \)
only when \( \ (x+2)^2 \ (x-3) = 0 \)
... when \( x = -2 \) (twice), \( x = 3 \)

What happens to \( y=P(x) \) when \( x \to \infty \) (increases towards \( \infty \))? Consider that \( (x + 2)^2 \) and \( (x - 3) \) both grow unboundedly large, both positive, as \( x \to \infty \). So \( (\frac{1}{2}) \ (x + 2)^2 \ (x - 3) \to \infty \) as \( x \to \infty \).

(i.e. grows unboundedly large)

... And when \( x \to -\infty \), \( (x + 2)^2 \) & \( (x - 3) \) both grow unboundedly large, but \( (x-3) \) is large negative, so the product is large & negative. \( (\frac{1}{2}) \ (x + 2)^2 \ (x - 3) \to -\infty \) as \( x \to -\infty \).

Plotting a few additional points, such as (-5,-36) & (-4,-14) & (1,-9) & (2,-8) & (4,18) adds accuracy & serves as a check on the analysis.
Sketch the graph of $y = \frac{4 - 2x}{3 - x}$ Label all the intercepts & asymptotes.

Rational function, therefore:
Domain: $y$ is defined for all values except where denominator = 0: $x = 3$

As $x$ approaches 3, $4 - 2x \to -2$,

while $3 - x \to 0$,

so the quotient $\to \pm \infty$

Thus there is a vertical asymptote at $x = 3$.**

As $x \to \infty$ $y \to 2$
As $x \to -\infty$ $y \to 2$

Thus the function has a horizontal asymptote* at $y = 3$.

y-intercept: when $x = 0$
$y = \frac{4}{3}$

x-intercept: when $x = 2$, $y = 0$

** More on the Vertical Asymptote:

As $x$ approaches 3 from below e.g. take $x = 2.5, 2.8, 2.9, 2.99, 2.9999 \ldots \to 3$
y grows unboundedly large, negative:
$y = \frac{-1}{2}, \frac{-1.6}{2}, \frac{-1.8}{2}, \frac{-1.98}{2}, \frac{-1.9998}{2}$

Similarly, computing function values for $x$ that are just above 3, demonstrates that the function values are positive and grow unboundedly large ($\to \infty$) as $x$-values approach 3 from above.

*Another way to discover the horizontal asymptote:

...divide & conquer:

$$\frac{-2x + 4}{-x + 3} = \frac{2x - 4}{x - 3} = 2 + \frac{2}{x - 3}$$

As $x \to \infty$, the latter portion $\to 0$

& $f(x) \to 2 + 0$ as $x \to \infty$