1. For \( f(x) = x^2 + 2x - 5 \), find \( \frac{f(x+h) - f(x)}{h} \) and simplify completely.

\[
f(x+h) - f(x) = \frac{(x+h)^2 + 2(x+h) - 5 - (x^2 + 2x - 5)}{h}
\]
\[
= \frac{x^2 + 2xh + h^2 + 2x + 2h - 5 - (x^2 + 2x - 5)}{h}
\]
\[
= \frac{x^2 + 2xh + h^2 + 2x + 2h - 5 - (x^2 + 2x - 5)}{h}
\]
\[
= \frac{2xh + h^2 + 2h}{h}
\]
\[
= \frac{h(2x + h + 2)}{h}
\]
\[
= 2x + h + 2
\]

**NOTE:**
- \( f(x+h) \) is NOT \( f(x)+h \)!
- \( f(x+h) \) is NOT \( f(x)\) \((x+h)\)!

2. Use the graph of the function at right to answer the following questions.

a. What is \( f(-2) \)? \( f(-2) = 2 \) (See \( \bullet \) on graph.)

b. What is the domain of \( f \)? \([-6, 4.5]\) (estimated)

c. What is the range of \( f \)? \([-4, 3]\)

d. On what interval(s) is \( f \) increasing?
   - \( f \) increases when \(-6 \leq x \leq -3 \) & \( 1.2 \leq x \leq 3 \) (roughly)
   - On the interval \([-6, -3]\) & on \([-1.2, ~3]\)

e. What is the average rate of change of \( f \) on the interval \([-3, 4]\)?

The height of the graph at \( x = -3 \) is \( y = 3 \). The height at \( x = 4 \) is \( y = -1 \). (See \( \bullet \) points on graph.)

So while \( x \) increased 7 units, \( y \) dropped from 3 to -1 \((3 - -1 = 4)\).

The slope of the line connecting the two points is thus:

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 3}{4 - (-3)} = \frac{-4}{7}
\]
3. Sketch the graph of \( y = -2(x+1)^3 \) using transformations of a familiar function.

Adding 1 to the argument shifts the graph 1 unit left. Multiplying by -1 “flips” the curve about the x-axis. Multiplying by 2 stretches it vertically.

4. Sketch the graph of \( f(x) = 2x^2 + 6x + 3 \). Label the coordinates of the vertex & y-intercept. Does this function have a maximum or minimum value? What is it?

\[
f(x) = 2(x^2 + 3x) + 3 = 2(x + ?)^2 + 3
\]

This is clearly the squaring function... Shifted left \( \frac{3}{2} \), stretched by the factor 2, shifted down \( \frac{3}{2} \). Thus \( -\frac{3}{2} \) is the minimum value.

ANOTHER WAY to reach this conclusion:
An alternative is to know that \( f(x) = ax^2 + bx + c \) has its extreme point at \(-b/2a\), which in this case is \(-6/4 = -\frac{3}{2}\).

The leading coefficient in \( f(x) = 2x^2 + 6x + 3 \), which is \(+2\), informs us the parabola opens upward, so we know the extreme point is a minimum.

Finally we evaluate \( f(x) \) at \( x = \frac{3}{2} \):

\[
f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 + 6\left(\frac{3}{2}\right) + 3 = -\frac{3}{2}
\]

... to obtain the minimum value.
5. Sketch the graph of the function \( f(x) = x^2(x-2)(x+1) \). Label all intercepts with their coordinates, and describe the "end behavior" of \( f \).

That \( f(x) \) is a 4\(^{th}\)-degree POLYNOMIAL* function is clear without computing: \( f(x) = x^4 - x^3 - 2x^2 \).

The ROOTS of \( f \) are 0, 0, 2, and -1.

\[
\begin{array}{c|cccccc}
\text{f(x)} & + & 0 & - & 0 & - & 0 & + \\
\hline
-1 & 0 & 2
\end{array}
\]

and when \( x \) is large (+ or -), \( f(x) \) is dominated by, and behaves like \( x^4 \), with \( y \) rising towards infinity.

* Therefore, the graph of \( f \) is smooth, continuous, and behaves like its largest power as \( x \to \infty \).

6. Sketch the graph of \( f(x) = \frac{2x}{x^2 - 1} = \frac{2x}{(x+1)(x-1)} \).

Domain: all reals except -1 & 1

RATIONAL function with vertical asymptotes at \( x=1 \) and \( x=-1 \).

\( f(0) = 0 \) .... Thus (0,0) is both x-intercept and y-intercept. There are no more intercepts.

As \( x \to \infty \), denominator (degree 2) grows much faster than numerator (degree 1); function behaves like \( \frac{2}{x} \), which we know \( \to 0 \) as \( x \to \infty \). So \( y=0 \) is horizontal asymptote.

It helps a lot to determine whether \( f(x) \) is positive or negative on each interval between points of interest.

\[
\begin{array}{c|cccc}
\text{f(x)} & - & ! & + & 0 & - & ! & + \\
\hline
-1 & 0 & 1
\end{array}
\]

One final comment on this: \( F \) is an ODD function; note the symmetry.
7. Solve the inequality, and express your answer in INTERVAL notation.

\[
\frac{5}{x + 3} \geq \frac{3}{x - 1}
\]

Q: 

\[
\begin{array}{cccc}
-3 & + & 1 & - & 7 \\
\end{array}
\]

\[
\frac{5}{x + 3} - \frac{3}{x - 1} \geq 0
\]

Q is + for all \(x\) between -3 & 1 and above 7.

Noting that the fractions do not exist at -3 & 1 eliminates -3 & 1 as solutions.

Testing 7, we get 0 ≥ 0, so 7 is a solution.

\[
\frac{2x - 14}{(x + 3)(x - 1)} \geq 0
\]

critical values (where the signs of the factors can change) are 7, -3, 1.

The solution set for this inequality is:

\((-3, 1) \cup [7, \infty)\)

8. For \(P(x) = 2x^3 + 5x^2 - 15x + 6\),

a. List ALL the possible rational zeroes of \(P(x)\).

\(P(x)\) has all integer coefficients,
so any rational roots must be of the form \(\frac{p}{q}\), where \(p\) divides 6 and \(q\) divides 2.

The possibilities consist of all viable combinations of

\[\pm 6, 3, 2, 1\]...which gives us the LIST: \(\pm 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}\) (Notice there are twelve!)

b. Use synthetic division to show that \(\frac{1}{2}\) is a zero of \(P(x)\).

\[
\begin{array}{c|cccc}
\frac{1}{2} & 2 & 5 & -15 & 6 \\
& & 1 & 3 & -6 \\
--- & --- & --- & --- & --- \\
2 & 6 & -12 & 0 \\
\end{array}
\]

The zero remainder tells us \(x - \frac{1}{2}\) is a FACTOR of \(P(x)\).

Thus \(\frac{1}{2}\) is a ROOT (aka “ZERO”) of \(P(x)\).

c. Find ALL the zeroes of \(P(x)\). Simplify your answers.

To find the remaining zeroes of \(P(x)\), we use the quotient, \(Q(x) = 2x^2 + 6x - 12\).
Aside from \(x = \frac{1}{2}\), only values of \(x\) that make \(Q(x) = 0\) will make \(P(x) = 0\).

\[2x^2 + 6x - 12 = 0\]
\[x^2 + 3x - 6 = 0\]...Does not factor. So we resort to the useful quadratic formula.

\[
x = \frac{-3 \pm \sqrt{9 - 4(-6)}}{2} = \frac{-3 \pm \sqrt{33}}{2}
\]

The third zero is the one we verified in part b: \(\frac{1}{2}\)
9. Find a fourth-degree polynomial \( P(x) = ax^4 + bx^3 + cx^2 + dx + e \) with REAL coefficients, and with 0 a root (zero) of multiplicity 2, and with 1+i a root or zero.

\( (x - 0) \) is a factor ... twice, and \( (x - (1+i)) \) is a factor and \( (x - (1- i)) \) is a factor.

Since any number \( r \) is a root of \( P \) if and only if \( (x - r) \) is a factor of \( P \) of multiplicity \( k \)

So \( P(x) = (x -0) (x-0) (x - (1+i)) (x - (1- i)) \)

**See Details below.

\[ \begin{align*}
&= x^2 ( x^2 - 2x + 2) \\
&= x^4 - 2x^3 + 2x^2
\end{align*} \]

(Note: The given specifications do not fully determine \( P \). Any polynomial of the form \( P(x) = A (x^4 - 2x^3 + 2x^2) \) would answer the request.)

**Details: \( (x - A) (x - B) = x^2 + (- A - B)x + AB \) ...for any numbers A & B

Here, \( A = 1 + i \) and \( B = 1 - i \),

so \( - A - B = -1 - i -1 + i = -2 \)

and \( AB = (1 + i)(1 - i) \)

\[ \begin{align*}
&= 1 + i - i - i^2 \\
&= 1 + 1
\end{align*} \]

10. A rectangular storage space is to be enclosed with 200 yards of fencing. One side of the storage space faces an existing fenced yard and does not require fencing. Let \( x \) be the length of the side that projects out from the existing fance.

a. Express the area of the storage space in terms of \( x \).

\[ \text{AREA} = \text{length} \times \text{width} \]

\[ A(x) = (200 - 2x) x = 2(100 -x) x \]

If \( x = 60 \) yd, then 120 yd are used up by x-fences, so “y” can be only 200 – 120 yd.

If \( x = a \) yd, then 2a yd are used up by x-fences, so only 200 - 2a remain for “y”.

b. For what value of \( x \) will the area be maximum?

We see the function above is a parabola, opening down, with x-intercepts at 0 and 100. Therefore, the maximum value for \( A \) must occur when \( x \) is midway between these zeroes. ... when \( x = \boxed{50} \) (yd)

Alternatively, we can observe \( A(x) = -2x^2 + 200x \) and complete the square or use the formula to obtain \( x = -200/4 \).

(Not requested, but the maximum area is 50 yd•100yd = 5000 square yards.)