❦ *Extra Practice for Chapter 6* ❧

1. *Graph an exponential function.*

   a. Sketch the graph of $f(x) = 2 - e^{-x}$. Label any intercepts and linear asymptotes.

   b. Sketch the graphs of $y = 3^x$ and $y = (⅓)^x$ and $y = 3^{x+1}$ and $y = 3^{-x}$ on one set of coordinate axes.

2. *Graph a logarithmic function.*

   a. Sketch the graph of $f(x) = \ln(x-2) + 3$. Label any intercepts and asymptotes.

   b. Sketch the graph of $g(x) = \ln|x-1|$

3. *Simplify an expression using properties of logarithms.*

   a. Simplify completely: $2\log_5(5x) + \log_5(5x) - 3\log_5 x$

   b. Simplify: $(\log_2 16 - \log_4 16)$ and $\dfrac{\log_4 6789}{\log_2 6789}$

   c. Simplify: $\log_2(1/4) + \log_2 8 - \log_2(3) + \log_2(9) + \log_2(.5)$

4. *Solve a logarithmic equation.*

   a. Solve $\log_5 x - \log_5(x-20) = 1$

   b. Solve $2\log|x| - \log(x+2) = 0$

5. *Solve an exponential equation.*

   a. Solve $2^{x+1} = 5^x$ exactly. (Give exact answer, not a decimal approximation.)

   b. Solve $3^x = 9^{x+1}$

   c. Solve $2^x = 3$

6. *Solve a problem involving exponential growth.*

   a. A bacteria population of 2400 has a relative growth rate of 4% per hour. How long will it be until the population reaches 14,400?

   b. A bacteria population had a count of 24,000 at 1 hour, and a count of 96,000 at 2 hours. Find an expression for the population at t hours.

   c. If you deposit $5000 into an account earning interest at the annual rate of 5%, compounded annually, how long will it take to triple? ... what if compounded continuously? (This is from §6.7, periodic compounding, not covered.)

7. *Identify 1-1 functions and find inverses.*

   Which of the following functions is one-to-one. If the function is 1-1, find its inverse.

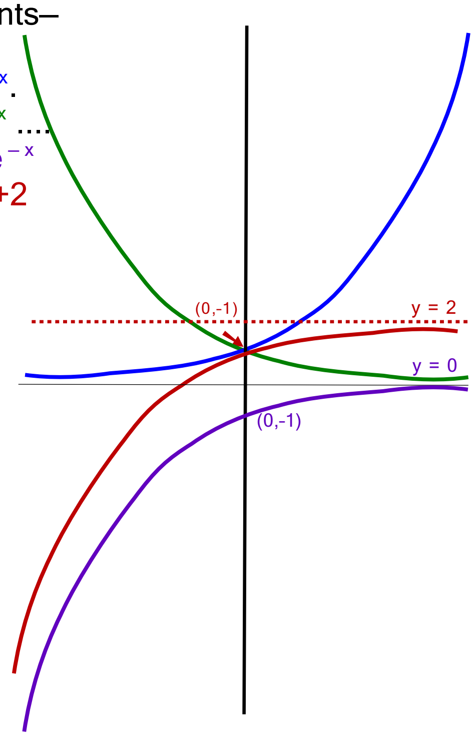
   a. $f(x) = 5x^6 - 3x + 77$

   b. $g(x) = \dfrac{1}{1 - x^2}$

   c. $h(x) = 1 - \sqrt{x - 1}$

   d. $k(x) = \dfrac{3x + 5}{2x - 1}$

1a. If we know the graph of $f(x) = a^x$ (for $a > 1$), plotting a few points– (-1, 1/e) and (0,1) and (1, e) (Keeping in mind that $e \approx 2.71$)– should give us a reasonably accurate representation of $y = e^x$.
From that, reflection through the y-axis gives us $y = e^{-x}$....
From the above, reflection through the x-axis gives us $y = -e^{-x}$
Finally, shifting the graph of $y = -e^{-x}$ up two units: $y = -e^{-x} + 2$
we have the desired graph: $y = 2 - e^{-x}$.

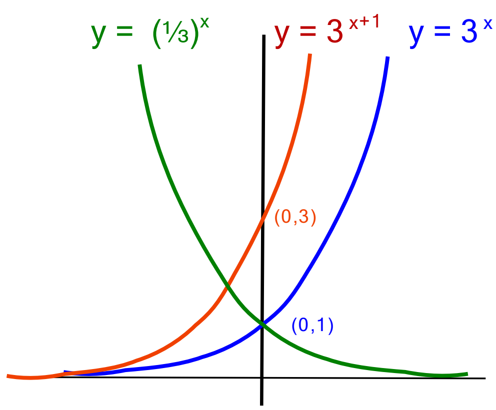


as $x \to \infty$, $-e^{-x} \to 0$
consider, e.g. $f(100) = -e^{-100} = -\dfrac{1}{e^{100}}$

So curve $y = -e^{-x}$ has asymptote $y = 0$.
and thus $y = 2 - e^{-x}$ has asymptote $y = 2$.

$e^0 = 1 \to (0, 1)$ is the y-intercept of $y = e^x$ (& $y=e^{-x}$)
$-e^{-0} = -1 \to (0, -1)$ is y-intercept for $y = -e^{-x}$
$2 - e^{-0} = -1+2 \to (0, 1)$ is y-intercept for $y = 2 - e^{-x}$

1b. $y = 3^x$ is quite close to $y= e^x$. We plot at least the points (-1, 1/3) and (0,1) and (1, 3), and sketch the exponential curve.

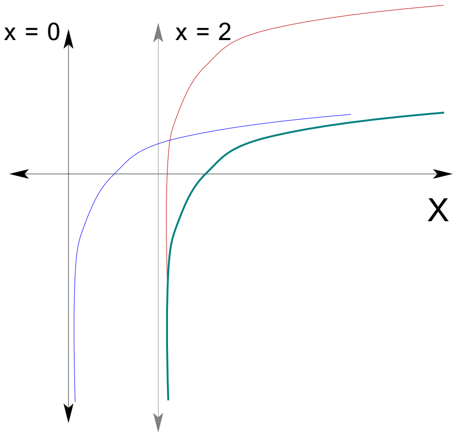


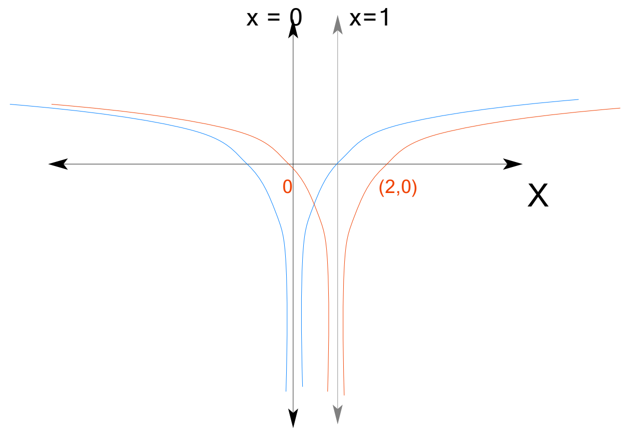
$y = (1/3)^x = (3^{-1})^x = 3^{-x}$

These two functions, $y = (1/3)^x$ and $y = 3^{-x}$ are identical.
$y = 3^{-x}$ is obviously the reflection of $y = 3^x$ through the y-axis.

$y = 3^{x+1}$ can be viewed a variety of ways:
$y = 3^{x+1} = \exp_3(x+1)$, which is the graph of $y = \exp_3(x)$, shifted one unit to the left.
$y = 3^{x+1} = 3^x\, 3^1$, which is the graph of $y=3^x$, multiplied by 3, thus stretched vertically by factor 3.

2. The graph of $y = \ln(x-2) + 3$ is the same as that of $y = \ln x$, but shifted 2 units to the right, $\ln(x-2)$, so its domain is $(2,\infty)$ rather than $(0,\infty)$, then moved vertically 3 units (because y IS 3 plus $\ln(x-2)$).





Where x is positive, the graph of $y = \ln|x|$ is identical with $y = \ln x$.
Where x is negative, $\ln|x|$ is just the reflection of $y = \ln x$ through the y-axis.
Then $y = \ln|x-1|$ is the graph of $y = \ln|x|$, shifted 1 unit to the right, so the vertical asymptote occurs at x = 1, rather than at x = 0.

3. Simplify an expression using properties of logarithms.

a. Simplify completely: $2\log_5(5x) + \log_5(5x) - 3\log_5 x$

(Just getting space to think!) ➙ $2\log_5(5x) \;+\; \log_5(5x) \;-\; 3\log_5 x$

Want to use the log properties $\log_5(5x)^2 + \log_5(5x) - \log_5 x^3$

But the 2 & 3 are "in the way" $\log_5 \dfrac{(5x)^2\,(5x)}{x^3}$

so we use $\log x^a = a \log x$

then reduce! $\log_5 \dfrac{5^2 x^2\, 5x}{x^3} = \log_5 5^3 = 3$

As with most simplifications, there are many ways to approach this...

b. Simplify: $(\log_2 16 - \log_4 16) = 4 - 2 = 2$

b. Simplify: $\dfrac{\log_4 6789}{\log_2 6789} = ½$

If you think about this the right way, the answer is obvious, ½.
But here's a METHOD:

$$\frac{\log_4 6789}{\log_2 6789} = \frac{\dfrac{\log_2 6789}{\log_2 4}}{\log_2 6789} = \frac{\dfrac{1}{\log_2 4}}{1} = ½$$

c. Simplify:

$\log_2(1/4) + \log_2 8 - \log_2(3) + \log_2(9) + \log_2(.5)$

$-2 + 3 + \log_2 \dfrac{9}{3} + -1 = \log_2 3 = \dfrac{\ln 3}{\ln 2}$

¿ $\log_2 3$?

$\log_2 3 = Y$
where $2^Y = 3$ ...Take $\ln$ of both sides)
$\ln 2^Y = \ln 3$ ...Use $\log x^a = a \log x$
$Y \ln 2 = \ln 3$ ...Solve for Y....

4a. Solve a logarithmic equation.

a. $\log_5 x - \log_5 (x-20) = 1$

$\log_5 \frac{x}{x-20} = 1$

$\frac{x}{x-20} = 5$

$x = 5x - 100$

$x = 25$

$\log \frac{A}{B} = \log A - \log B$

Exp base 5 undoes log base 5, and...
if $Q = T$, then of course $5^Q = 5^T$

Check (required!): $\log_5 25 - \log_5 (25-20) = 1$ is true!

4b. b. Solve $2\log|x| - \log(x+2) = 0$

$\log x^2 - \log(x+2) = 0$

$\log \frac{x^2}{x+2} = 0$

$\frac{x^2}{x+2} = 1$ ✳

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$, or -1

log a - log b = log a/b

If log W = 0 then W must be 1.

Assuming $x + 2 \neq 0$, we multiply by x+2,

(Alternatively, we could subtract 1 from both sides of ✳, then simplify the left side to one fraction, and end up solving this same statement!)

Test both of these: 2 log2 - log 4 = 0 ✓
2 log1 - log 1 = 0 ✓

5. Solve an exponential equation.

a. Solve (answer must be exact): $2^{x+1} = 5^x$

$2^{x+1} = 5^x$

$\ln(2^{x+1}) = \ln(5^x)$

$(x+1)\ln 2 = x \ln 5$

$x \ln 2 + \ln 2 = x \ln 5$

$\ln 2 = x(\ln 5 - \ln 2)$

$\frac{\ln 2}{\ln 5 - \ln 2} = x$

To solve exponential; haul out the logs

and use those log properties!

Hey, $\ln 2$ and $\ln 5$ are just numbers !

(Getting all the x's on ONE side....)

b. $3^x = 9^{x+1}$

$\ln 3^x = \ln 9^{x+1}$

$x \ln 3 = (x+1) \ln 9$

$x \ln 3 = (x+1) \ln 3^2$

$x \ln 3 = (x+1)\, 2 \ln 3$

$x = 2(x+1)$

$x = 2x + 2$

$x = -2$

method one:

S.O.P. Use logs.

"Cancel" (divide out) those $\ln 3$ factors!

method two:

$3^x = 9^{x+1}$

$3^x = (3^2)^{x+1}$

$3^x = 3^{2x+2}$

$x = 2x + 2 \rightarrow x = -2$

This second method may be used only because of the simple relationship between 3 & 9. The first method always works. Since the exponential function 3^x is 1-1: if $3^a = 3^b$, a must = b.

c. Solve: $2^x = 3$

$\ln 2^x = \ln 3$

$x \ln 2 = \ln 3$

$x = \boxed{\dfrac{\ln 3}{\ln 2}}$ aka $\log_2 3$ ... what a surprise! [hey! If $2^x = 3$, then x must be the power needed on 2 to make 3.... !!!]

6. Solve a problem involving exponential growth.

a. A bacterial population of 2400 has a relative growth rate of 4% per hour. How long will it be until the population reaches 14,400?

Hmmm.... after 1 hour, there will be 2400(1+.04);
after two hours, the number should be 2400(1+.04)•(1+.04) again
After t hours, the number of bacteria will be $2400(1.04)^t$.
So this question asks at what value of t will do this:

$2400(1.04)^t = 14400$

$(1.04)^t = 14400/2400 = 6$

$\ln (1.04)^t = \ln 6$

$t \ln (1.04) = \ln 6$

$t = \ln 6 / \ln 1.04 =$ 45.68 (hours)

If you assume continuous compounding,
you get $A(t) = 2400\, e^{.04t}$
and the ultimate solution here is
$t = \ln 6/.04 \approx 44.79$ (hours)
[Not Much Difference!]

b. A bacteria population had a count of 24,000 at 1 hour, and a count of 96,000 at 2 hours. Find an expression for the population at t hours.

For maximum simplicity, we assume continuous compounding.... Then the population at time t would be $A(t) = A_o e^{rt}$

$24K = A_o e^{r1}$

$96K = A_o e^{r2}$ $\rightarrow \dfrac{A_o e^{r2}}{A_o e^{r1}} = \dfrac{96}{24} = 4$

$\rightarrow e^r = 4$

and $A(t) = A_o e^{rt} = A_o 4^t$
And then A_o must be 6000,
because we know $24000 = A_o \cdot 4^1$

$A(t) = A_o e^{rt} = 6000\,(4^t)$

I also did this problem just thinking about interest:

In one hour, the pop. went from 24K to 96K, an increase of 72K, which is an increase of 300%. So this works like "compound interest" with an interest rate (hourly, not annual) of 300%....
$A(t) = A_o(1+3.00)^t = A_o 4^t$, where t = # of hours elapsed.
...proving once again that money helps!

c. If you deposit \$5000 into an account earning interest at the annual rate of 5%, compounded annually, how long will it take to triple? (This is from §6.7, not covered.)

TRANSLATION:

" ... how long will it take to triple? "
At what time t will the amount be 3 times the initial amount?
i.e. Solve for t: so:

(important! This is what takes the pain out of the "word problem".)

$$A(t) = 3A_o$$

$$A_o e^{rt} = 3A_o$$

$$\$5000\,(1 + .05)^t = \$15000$$

$$(1 + .05)^t = 3$$

$$\ln(1.05)^t = \ln 3$$

$$t \ln(1.05) = \ln 3$$

$$t = \ln 3 / \ln 1.05 \doteq 22.52 \text{ (years)}$$

How does answer to (b) change if the interest is compounded... daily? ...Continuously?

Daily: After t years the value is $\$5000\,(1 + .05/365)^{365t}$

So, in a manner similar to above, we solve:

$(1 + .05/365)^{365t} = 3$

And we get

$365t \cdot \ln (1 + .05/365)^{365t} = \ln 3$

$$t = \frac{\ln 3}{365 \cdot \ln (1 + .05/365)} \doteq 21.97 \text{ (years)}$$

With continuous compounding, the computation is much neater:

The balance after t years at annual rate "r" for initial deposit P is $P e^{rt}$ and we want that to be 3P.

We solve $Pe^{rt} = 3P$

(which is just) $e^{rt} = 3$

Taking $\ln$ of both sides: $rt = \ln 3$ .... $t = (\ln 3) \div r$

In this case r = .05/yr, so $t = \dfrac{\ln 3}{.05/\text{yr}} \doteq 21.97$ yr (...OK, 22 years)

7. Identify which of the following functions is one-to-one. If the function is 1-1, find its inverse.

a. $f(x) = 5x^6 - 3x + 77$

One way to determine that this function is not one-to-one is to consider its graph– you know the graph of f must resemble that of $y = 5x^6$ for large (both large – and large +) values of x.

b. $g(x) = \dfrac{1}{1 + x^2}$

Here ask yourself how the value of x affects the outcome g(x).... the first thing that happens is the squaring of x. After that x never enters into the computation– only x^2 is used.
Now squaring x obliterates the difference between + and – x-values; that is, $(-a)^2$ & $(a)^2$ are the same. So g(–3) = g(3)... and that is all we need to know to say g is NOT one-to-one.

c. $h(x) = 1 - \sqrt{x - 1}$ Domain $[1,\infty)$ Range $(-\infty, 1]$

If you know the shape of the graph (and you should!), then you know h is 1-1.
But suppose you don't think of that simple approach....
Then you could try to solve for the inverse (see method used in part d)....
Or you could use this shortcut to find the inverse:

h takes x and	. . .So h^{-1} must	
SUBTRACTS 1,	subtract 1	$x-1$
then takes SQRT of that,	Switch the sign,	$-(x-1) = 1 - x$
then switches the sign,	Square	$(1 - x)^2$
then adds 1.	& ADD 1.	$(1 - x)^2 + 1$

$h^{-1}(x) = (1 - x)^2 + 1$ or $x^2 - 2x + 2$
for $x \le 1$ only.

d. $k(x) = \dfrac{3x + 5}{2x - 1}$

You should know the graph of this linear rational function resembles the graph of y = 1/x, and so is 1-to-1. But even without knowing the function is 1-1 we can attempt to find the inverse, using this fairly standard procedure:

Starting with $y = \dfrac{3x + 5}{2x - 1}$, we first interchange x & y (since that's what the inverse does!)

$x = \dfrac{3y + 5}{2y - 1}$ and solve for y !

(If you need help here, pretend x is 4.)

$x(2y - 1) = 3y + 5$

$2xy - x = 3y + 5$

$2xy - 3y = x + 5$

$y(2x - 3) = x + 5$

$y = \dfrac{x + 5}{2x - 3}$ State result using the functional notation, $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x + 5}{2x - 3}$$