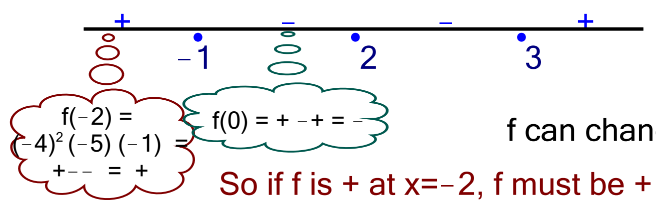
❦ *Detailed Solutions to Essential Skills for Chapter 5* ❧

1. (§5.1) Sketch the graph of $f(x) = (x-2)^2 (x-3)(x+1)$ $= x^4$ + lower terms....

2 is a root or zero of multiplicity two, so we expect a “bounce” of the graph at (2,0).
3 is a simple root or zero, so we expect the function to pass through the x-axis at (3,0).
−1 is a simple root or zero, so we expect the function to pass through the x-axis at (−1,0).
Our signs analysis will verify these predictions.

When x is −1 2 3 f(x) is 0.
So I mark those “0" or place a dot to remind me the points (−1,0) & (2,0) & (3,0) belong to f.

f(x) is ....
when x is



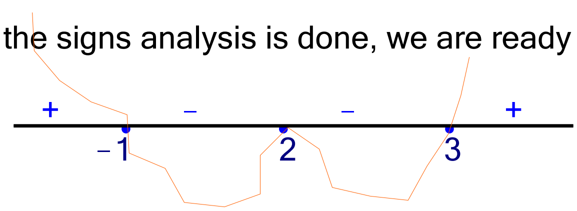
f can change sign ONLY at the CVS...

So if f is + at x=−2, f must be + at all x less than −1, the first CV.

Similarly, if f(0) is negative, then f must be negative at all x between −1 and 2.

Et cetera. Once the signs analysis is done, we are ready to make a very rough sketch:

f(x) is ....
when x is



OK, maybe that’s a little too rough.

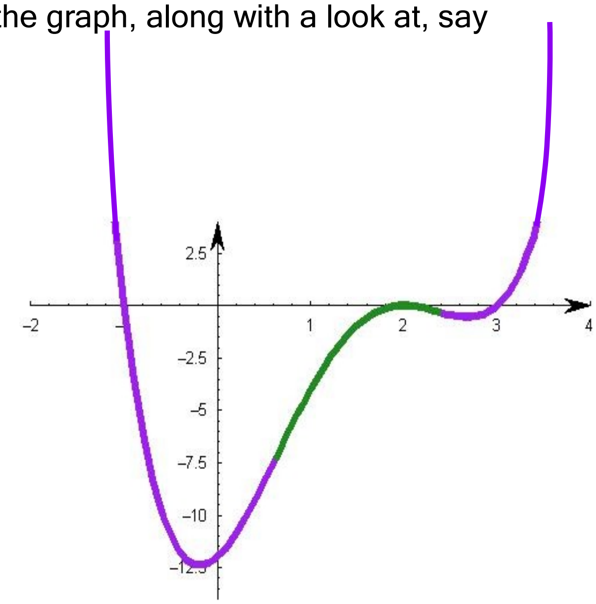
After all, we know this should be a continuous, SMOOTH curve!

Since there is a minimum value between −1 and 2, and another between 2 and 3, we might do a little computation, to see how low the graph falls. We need the y-intercept anyway.

$f(0) = (-2)^2 (-3)(1) = -12$
$f(1) = (-1)^2 (-2)(2) = -4$
$f(2.5) = (.5)^2 (-.5)(3.5) = -3.5/6 = -7/16 = -.4375$
$f(2.7) = (.7)^2 (-.3)(3.7) = -.5439$

So, now, placing some of these additional points on the graph, along with a look at, say
$f(-2) = (-4)^2 (-5)(-1) = 90$
and
$f(4) = (2)^2 (1)(5) = 20$
$f(5) = (3)^2 (2)(6) = 108$

lets us know the graph looks pretty much like this:



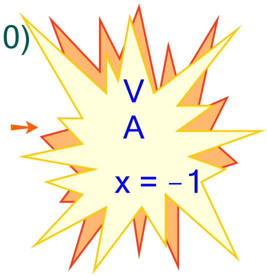
Notice this is consistent with the fact that seen from afar, this graph should resemble that of $y = x^4$ (because x^4 is the leading term, and dwarfs the rest when x is very large, negative or positive).

(In fact the lowest point occurs just before 0, and the other, local, minimum occurs just before 3. But we cannot be expected to find that with the tools we have in this course.
The x-coordinates of these points are $(-5 \pm \sqrt{33})/4$.)

2. Sketch the graph of $R(x) = \frac{x^2 - x - 12}{x + 1}$ $\quad \frac{P(x)}{Q(x)}$

$R(0) = \frac{0^2 - 0 - 12}{0 + 1} = -12$ → y-intercept is (0, −12)

R(x) = 0 only when P(x) = 0 $\quad x^2 - x - 12 = 0$, $(x+3)(x-4) = 0$ → x-intercept s are (−3, 0) & (4, 0)



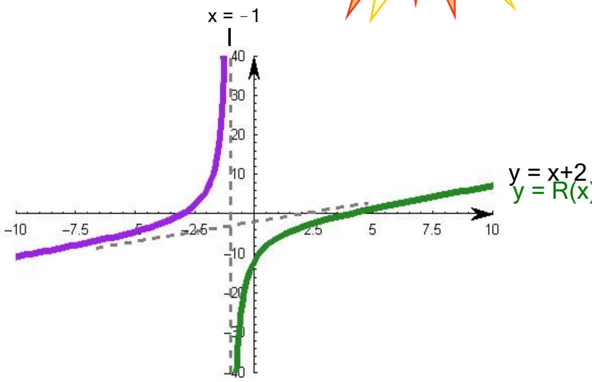
Domain: all reals except where x+1 = 0 ... i.e. all reals except −1.
What happens near x = −1 ? Numerator → −10 while Denominator → 0

Horizontal asymptote? NO!.
Clue #1: Degree numerator > degree denominator
Clue #2: Divide & conquer!!

−1	1	−1	−12
		−1	2
	1	−2	−10

So $\frac{x^2 - x - 12}{x + 1} = x - 2 + \frac{-10}{x + 1}$

i.e. f(x) = line function + hyperbolic function
y = x+2 $\quad$ y = 1/x, shifted....



As $x \to \infty$, the value of f(x) gets ever closer to the line y = x - 2 because $-10/(x+1) \to 0$ as $x \to \infty$.
That is to say, f has a linear asymptote, the line y = x - 2. (an oblique asymptote).

3. Solve $\frac{x}{x+2} < \frac{1}{x}$ Compare to zero. Resist the temptation to multiply by x and x+2 ... or to "cross multiply" (even if the Coach said so!!)

$\frac{x}{x+2} - \frac{1}{x} < 0$ Now simplify. Add the fractions... for which a common denominator helps.

$\frac{x \cdot x}{(x+2) \cdot x} - \frac{1(x+2)}{x(x+2)} < 0$ (Giving the fractions common denominators.)

$\frac{x^2 - x - 2}{x(x+2)} < 0$ After combining the fractions, the LHS can be factored to reveal

$\frac{(x-2)(x+1)}{x(x+2)} < 0$ critical values are 2 , −1 , 0 , −2 .

Q: + − + − + (signs on the intervals separated by −2, −1, 0, 2)

We check out the signs on each interval and find the intervals where Q < 0 , − , then check the endpoints, and state the solution set:

(−2, −1) U (0, 2)

4. Find all the roots/zeros of $P(x) = 2x^3 - 5x^2 + 6x - 2$

We observe there are no common factors to extract.
No obvious factorization. Look for a root among the rationals: ± 2, 1, ½ are candidates.

½	2	−5	6	−2
		1	−2	2
	2	−4	4	0

→ $2x^3 - 5x^2 + 6x - 2 = (x - \tfrac{1}{2})(2x^2 - 4x + 4)$
$= (2x - 1)(x^2 - 2x + 2)$

... then use the quadratic formula to locate the last two roots:

$x^2 - 2x + 2 = 0$

when $x = \frac{+2 \pm \text{sqrt}(4 - 8)}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Zeros of P are: ½, 1+i , 1−i