# ❦ Extra Polynomial & Rational Practice ❧!
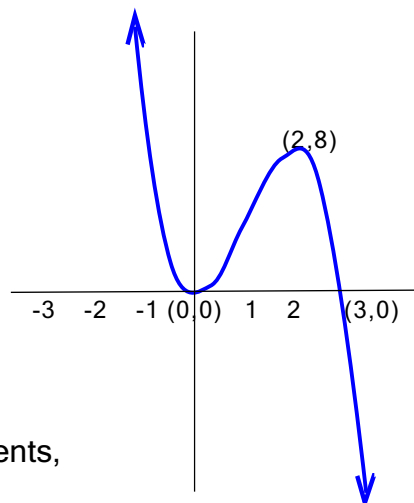
1.  Graph these polynomial functions.   Label all intercepts and describe the end behavior.

    a.    $P(x) = x^4 - 2x^3 - 15x^2$.

    b.    $P(x) = x^5 - 5x^3 + 4x$.

2.  Use polynomial long division.

    EG:    Divide ...    $\dfrac{6x^4 + 2x^2 + 22x}{2x^2 + 5}$

(2,8)

-3   -2   -1 (0,0)  1   2   (3,0)

3.  Construct a polynomial to specifications!

    a.    Find a second-degree polynomial with real coefficients,
          with zeroes $-3$ and 2, passing through  (4,3).

    b.    Find a polynomial whose  graph is be similar to that shown here ↗.

    c.    Find a polynomial of minimal degree, with real coefficients, having zeroes $2i$  and $1 - i$,
          and  constant term 16.

4.  Find all the zeros of these polynomial functions.

    a.    For  $P(x) = 5x^3 - 22x^2 + 18x - 4$,  list all the theoretically possible rational zeroes of P;
          use synthetic division to locate a zero; then find the remaining zeroes.

    b.    Find __all__ the zeroes of  $P(x) = 4x^5 + 15x^3 - 4x$

5.  Graph these rational functions.  Include the following:
    • Find the x and y intercepts.
    • Find the equation of any vertical asymptote.
    • Find the equation of any horizontal asymptote.
    • Sketch the graph.

    a.    $f(x) = \dfrac{3x - 4}{x - 2}$

    b.    $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

    c.    $f(x) = \dfrac{x^3 - 1}{x}$

    d.    $f(x) = \dfrac{x^2 - 2x - 15}{x^2 - 4}$

6.  Miscellaneous:

    a.  Find the value of $P(x) = 2x^5 - 20x^4 - 20x^3 - 20x^2 - 20x - 22$ without raising 11 to a power.

    b.  If $P(x) = (x - 2)(x^3 - 4x^2 + 7x + 13) + 7$ , what is P(2)?

# ❦ Solutions to Extra Polynomial & Rational Practice ❧

1. Graph a polynomial function. Label all intercepts and describe the end behavior.

  a. $P(x) = x^4 - 2x^3 - 15x^2$.

  (1) Domain = R, of course (since this is a polynomial function... domain must be R.)

  (2) y-intercept: $P(0) = 0^4 - 2 \cdot 0^3 - 15 \cdot 0^2 = 0$

  (3) x-intercepts: $x^4 - 2x^3 - 15x^2 = 0$        SIGNS: P(x) is
      $x^2 ( x^2 - 2x - 15) = 0$        when x is between
      $x^2 ( x - 5) (x + 3) = 0$
      x = 0 or 0 or 5 or -3

  SIGNS line:  $+ \quad {}_0 \quad - \quad {}_0 \quad - \quad {}_0 \quad +$
  when x is between   -3   0   5

  (4) the outer limits: As $x \to \infty$, $x^4$ dominates the other terms, so $f(x) \to \infty$.
      (i.e. f(x) grows unboundedly large)
      Likewise, as $x \to \infty$, $x^4$, dominating the other terms, takes $f(x) \to \infty$

  (5) ...the graph:
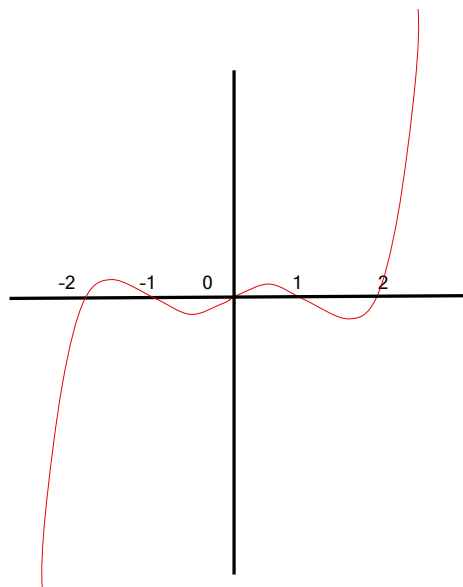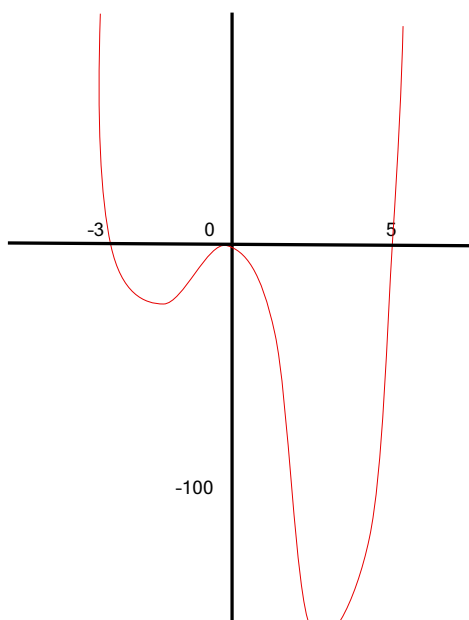


  b. $P(x) = x^5 - 5x^3 + 4x$.

  (1) Domain = ℝ

  (2) y-intercept: when x = 0, y = P(0) = 0

  (3) x-intercepts: y = 0 when $x^5 - 5x^3 + 4x = 0$        SIGNS: P(x) is
      when $x (x^4 - 5x^2 + 4) = 0$   when x is between
      $x (x^2 - 4)(x^2 - 1) = 0$
      $x (x+2)(x-2)(x+1)(x-1) = 0$
      x = 0,-2, 2, -1, 1

  SIGNS line:  $- \quad {}_0 \quad + \quad {}_0 \quad - \quad {}_0 \quad + \quad {}_0 \quad - \quad {}_0 \quad +$
  when x is between   -2   -1   0   1   2

  (4) the outer limits:        as $x \to \infty$, f(x), dominated by $x^5$, $\to \infty$
      ax $x \to -\infty$, f(x), again ruled by $x^5$, $\to -\infty$

  (5) graph above, right. P is an odd function.

      NOTE: The curve is smoother than this pathetic illustration indicates.
      It is just difficult to draw these curves with a mouse and flaky software.
      After the seventeenth try, patience vanishes.
      When I win the lottery, I will buy some fancy software.

2.  Use polynomial long division.

EG:   Divide ...   $\dfrac{6x^4 + 2x^2 + 22x}{2x^2 + 5}$

$$2x^2 + 5 \quad \overline{) \; 6x^4 + 0 + 2x^2 + 22x + 0} \qquad 3x^2 \quad - \;^{13}/_2$$

$$\begin{array}{r} 3x^2 \quad - \;^{13}/_2 \\ 2x^2 + 5 \;\overline{)\; 6x^4 + 0 + 2x^2 + 22x + 0} \\ \underline{6x^4 \qquad + 15x^2} \\ -13x^2 \\ \underline{-13x^2 \qquad - \;^{65}/_2} \\ 22x + \;^{65}/_2 \end{array}$$

SO   $\dfrac{6x^4 + 2x^2 + 22x}{2x^2 + 5} \;=\; 3x^2 \; - \;^{13}/_2 \; + \; \dfrac{22x + \;^{65}/_2}{2x^2 + 5}$

OR   $6x^4 + 2x^2 + 22x \;=\; (3x^2 - \;^{13}/_2)\,(2x^2 + 5) + 22x + \;^{65}/_2$

Notice the last statement shows the way to check your division results ($\rightarrow$ multiply & add) !!


3.  a.   Find a second-degree polynomial with real coefficients with zeroes $-3$ and 2, passing through (4,3).

   r is a zero of polynomial P if, and only if, (x – r) is a factor of P(x) .

   Having zeroes $-3$ and 2 requires that both (x – $-3$) and (x – 2) be factors.
   Since P must be a 2$^{nd}$-degree polynomial, the only other possible factor is a constant.
   So:

   P(x)  =  A (x + 3) (x – 2)   where A is some currently undetermined constant.

   The additional requirement that the curve contain (4,3) determines A:

   P(4)  =   A (4+3) (4 – 2)  =   A (7) (2)  = 14A
   But P(4) must be 3... so  14A must be 3.  therefore  A = 3/14 and
   P(x)  =  (3/14) (x + 3) (x – 2)  =   (3/14) ( $x^2$ + x – 6)


   b.   The graph shows us there is a repeated zero*, of even order, at x = 0, and a simple zero
   at  x = 3.  It also tells us P(2) must be 8.

   The zeroes inform us that (x-0)(x-0)(x-3) must be part of the factorization of P.
   Therefor, P(x) = x•x•(x–3)•(other factors).  The degree of P , which is 3,  tells us that the
   "other factors" can be only constant.

   Stated briefly, we know:
   Zeroes are 0, 0, and 3 & polynomial has degree 3, so
   P(x) = K $x^2$ (x–3)

   Since P(2) = 8:
   8 = P(2) = K•$2^2$(2–3) = -4K, so K must be $-2$.

   Therefore, P(x) =  $-2$ $x^2$ (x–3) , or P(x) =  $-2x^3$ + 6 $x^2$

c.    Find a fourth degree polynomial with real coefficients, with zeroes 2i and 1 – I, and constant term 16.

The complex zeroes of a polynomial with REAL coefficients must occur in conjugate pairs.

Since we are required to find a polynomial with real coefficients, and since 1 – I is a zero, its conjugate, 1 + I must also be a zero.  (Otherwise we'd end up with complex coefficients.)
Likewise, since 2i is a zero, – 2i must also be a zero.  Therefore:

$P(x) = A ( x - (1 + i) ) ( x - (1 - I) )  ( x - 2i )  ( x + 2i )$

$= A ( x^2 - ( 1+i + 1- I ) x + (1+i)(1- i) ) ( x^2 - 4i^2 )$

$= A ( x^2 - 2x + 2 ) ( x^2 + 4 )$

The additional requirement that the constant term be 16 determines A.
A(2)(4) must be 16.... thus A must be 2, and $P(x) = 2 (x^2 - 2x+ 2) ( x^2 + 4 )$
$= 2 (x^4 - 2x^3+ 6x^2 - 8x + 8)$
$= 2x^4 - 4x^3+ 12x^2 - 16x + 16$

4.    Find all the zeros of a polynomial function.

a.    For $P(x) = 5x^3 - 22x^2 + 18x - 4$,  list all the theoretically possible rational zeroes of P; and use synthetic division to locate one; then find the remaining zeroes.

RATIONAL zeroes of a polynomial with INTEGER coefficients must be of the form p/q where p divides the constant term and q divides the leading coefficient.

So any rational zeroes of $5x^3 - 22x^2 + 18x - 4$ must be of the form:
$\dfrac{\pm4 \text{ or } \pm2 \text{ or } \pm 1}{\pm5 \text{ or } \pm1}$    which yields:   $\pm1 \ \pm2 \ \pm4 \ \ \pm\dfrac{1}{5} \ \ \pm\dfrac{2}{5} \ \ \pm\dfrac{4}{5}$

TESTING we divide $5x^3 - 22x^2 + 18x - 4$ by x – r  for each potential zero r:

```
 1 | 5  -22   18   -4              -1 | 5  -22   18   -4               2 | 5  -22   18   -4
   |      5  -17    1                 |     -5   27                      |     10  -24  -12
     5  -17    1   -3 ☹                5  -27        ☹                     5  -12   -6   ☹

-2 | 5  -22   18   -4               4 | 5  -22   18   -4              -4 | 5  -22   18    -4
   |    -10  -17    1                 |     20   -8   40                 |    -10  128  -12
     5  -32    1   -3 ☹                5   -2   10   ☹                     5  -32        ☹

1/5 | 5  -22   18    -4            -1/5 | 5  -22   18   -4             2/5 | 5  -22   18   -4
    |      1  -21/5  69/25              |     -1  -23/5 ☹                  |      2   -8    4
      5  -21  69/5        ☹              5  -23  67/5  ☹                    5  -20   10    0
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finally !**

So $P(x) = (x - {}^2/_5)(5x^2 - 20x + 10) = (5x - 2)(x^2 - 4x + 2)$

For the remaining zeroes we solve:  $x^2 - 4x + 2 = 0$
... using the quadratic formula:    $x = \dfrac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

PS $P(-x) = - 5x^3 - 22x^2 - 18x - 4$, so, by Descartes' Rule of Signs, there are no negative real zeroes.
                                        (We could have saved time **here!!!)

b.    Find all the zeroes of  $P(x) = 4x^5 + 15x^3 - 4x$

$4x^5 + 15x^3 - 4x =$                    Notice you can factor this!

$x(4x^4 + 15x^2 - 4) =$                  ...and this!

$x (4x^2 - 1)(x^2 + 4) =$

$x(2x+1)(2x- 1)(x +2i)(x - 2i)$        So the zeroes of P  are: $0, - \frac{1}{2} , \frac{1}{2} , -2i , 2i$

5.  a.    $f(x) = \dfrac{3x - 4}{x - 2}$

- Find the x and y intercepts.
      $f(0) = 2$;  $f(x) = 0$ when $3x - 4 = 0$ ... $x = {}^4/_3$
- Find the equation of the vertical asymptote.
      Near $x = 2$,  numerator $\to 2$, denominator$\to 0$,
      so fraction $\to \pm\infty$ ... vertical asymptote at $x = 2$.
- Find the equation of the horizontal asymptote.
      As $x \to\infty$,  $(3x - 4)/(x - 2) \to 3$
      so  $y = 3$ is the horizontal asymptote.
- Division also shows us:

$\dfrac{3x - 4}{x - 2} = 3 + \dfrac{2}{x - 2}$, which would have to be ...

   $y=1/x$, shifted 2 units right, stretched vertically by factor 2,
   then shifted upwards 3 units.
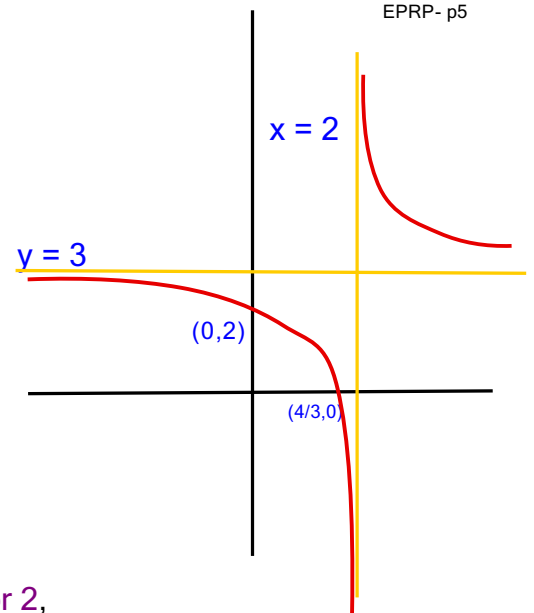
(In reality, curve has no "squiggles".  Mice!!!! )

SIGNS:          f(x) is     $\underline{\quad +\ _0\ -\ |\ +\quad}$
when x is between:                ${}^4/_3$    2

b.    $f(x) = \dfrac{x^2 - 3x - 4}{x - 2} = \dfrac{(x - 4)(x + 1)}{x - 2}$

- Intercepts:  $f(0) = 2$.
              $f(x) = 0$ when $x = 4, -1$.

- Vertical asymptote
      At  $x = 2$, for the usual reason.

- Horizontal asymptote
      None.  As $x\to\infty$ , $f(x) \to\infty$
      Divide to see non-horizontal asymptote:
      $\dfrac{x^2 - 3x - 4}{x - 2} = x - 1 + \dfrac{-6}{x - 2}$
      So we can see that $y = x - 1$ is an
      oblique linear asymptote for $y = f(x)$.
      (Since $-6/(x - 2)$ diminishes towards 0
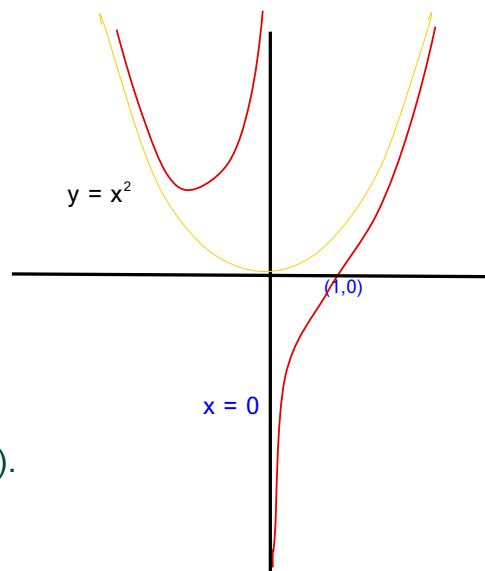       as x meanders out towards infinity.)

SIGNS:          f(x) is     $-\ _0\ +\ |\ -\ _0\ +$
when x is between          -1    2    4

$y = x - 1$
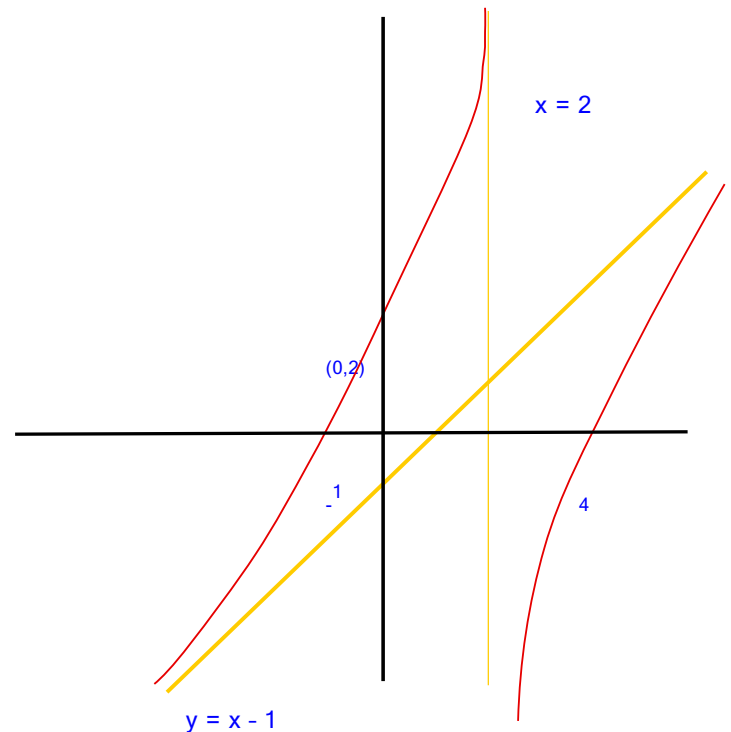
c.  $f(x) = \dfrac{x^3 - 1}{x} = \dfrac{(x-1)(x^2 + x + 1)}{x}$

x-intercept (1,0)**
vertical asymptote at x = 0

$f(x) = \dfrac{x^3 - 1}{x} = x^2 - \dfrac{1}{x}$

... this shows the given f is $x^2$ + an amount
that is very small when x is very large.  So
when x is large, the graph of y = f(x) is very
close to the graph of the familiar curve y = $x^2$.
And as x→ ∞,  the difference → zero.
This is asymptotic behavior (but not a linear asymptote).

NO worries: The only asymptotes you will be asked
to find are linear asymptotes!!!

$y = x^2$

x = 0

(1,0)

**of course, we determined that the remaining zeroes are complex, and thus not on the graph:

$x^2 + x + 1 = 0$        $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

d.  $f(x) = \dfrac{x^2 - 2x - 15}{x^2 - 4} = \dfrac{(x-5)(x+3)}{(x-2)(x+2)}$

← Here we see x =5, x = –3 make f(x) = 0 (x-int.)
← Here we see x = 2 , x = –2 cause 0 in the
domain, and, since the numerator is not 0 at
the same time, cause vertical asymptotes
VAs at  x = –2  &  x = 2

$f(0) = \dfrac{-15}{-4} = \dfrac{15}{4}$  is the y-intercept.

$f(x) = \dfrac{x^2 - 2x - 15}{x^2 - 4}$     Numerator is dominated by  $x^2$... behaves "like"  $x^2$ when x→∞
Denominator is dominated by  $x^2$... behaves "like"  $x^2$ when x→∞
So f(x) should behave "like"  $x^2/x^2$ when x→∞.
                              (Horizontal asymptote at y=1.)

$f(x) = \dfrac{1 - 2/x - 15/x^2}{1 - 4/x^2}$  ← Another way to reach that conclusion....

SIGNS:         f(x) is      + ₀ − | + | − ₀ +
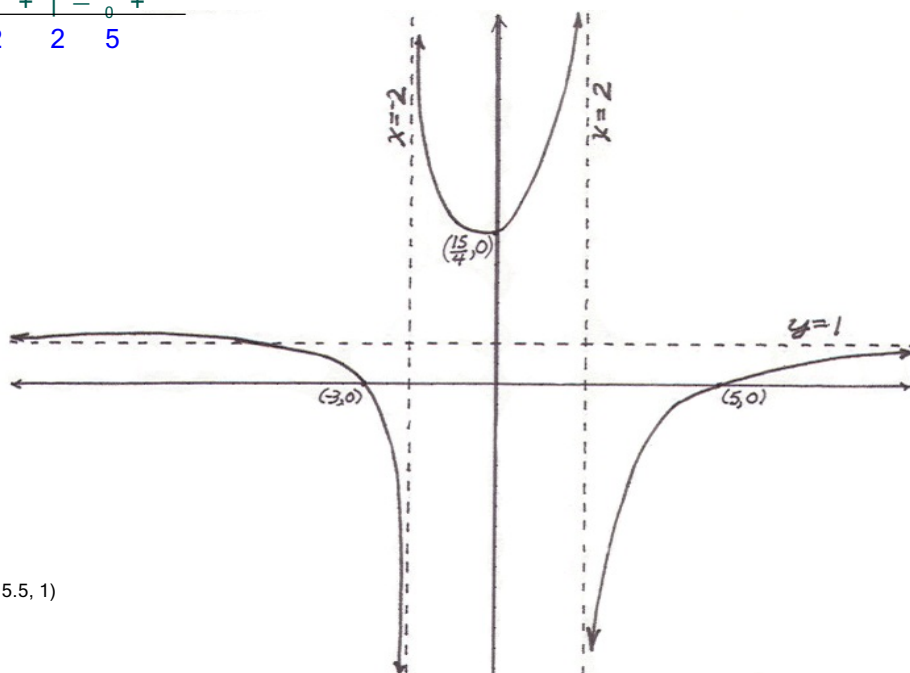when x is between          -3   -2   2   5

One final investigation helps us
determine an important feature
affecting the graph's shape.
Does the function
cross its horizontal asymptote?

Can    $\dfrac{x^2 - 2x - 15}{x^2 - 4} = 1$

We solve:
        $x^2 - 2x - 15 = x^2 - 4$
        $- 2x - 15 = -4$
        $-2x = 11$
        $x = -\dfrac{11}{2}$

And see that f crosses its horizontal asymptote at (x = –5.5, 1)

x = –2

x = 2

$\left(\frac{15}{4},0\right)$

y=1

(–3,0)

(5,0)

6.  The Remainder Theorem gives us the answers:
    If polynomial $P(x) = (x - c) Q(x) + R(x)$ where Q & R are the quotient and remainder
    polynomials for $P \div (x-c)$, then $R(x)$ is a constant, and $P(c)$ = that constant.

    a.  Find the value of $P(x) = 2x^5 - 20x^4 - 20x^3 - 20x^2 - 20x - 22$ @ 11
        – without raising 11 to a power.

    11 | 2  −20  −20  −20  −20  −22
       |      22   22   22   22   22
       ────────────────────────────
         2    2    2    2    2    0   This tells us that $P(11) = 0$, thus 11 is a zero, and $(x - 11)$ is a FACTOR of P.
                                       In fact, this tells us:   $P(x) = (x - 11)\ 2\ ( x^4 + x^3 + x^4 + x^3 + x + 1 )$

    b.  If $P(x) = (x- 2)(x^3 - 4x^2 + 7x + 13) + 7$ , what is $P(2)$?

        $P(2) = (2 - 2) ( whatever)   + 7$

            $= 0   + 7$

    SO:  $P(2) = 7$   ... JUST as the remainder theorem states.  (Go ahead, check it out!)