❦ Solutions to Extra Practice for Chapter 4 ❧

1. Graph a polynomial function. Label all intercepts and describe the end behavior.

a. $P(x) = x^4 - 2x^3 - 15x^2$.

(1) Domain = R, of course (since this is a polynomial function... domain must be R.)

(2) y-intercept: $P(0) = 0^4 - 2\cdot 0^3 - 15\cdot 0^2 = 0$

(3) x-intercepts: $x^4 - 2x^3 - 15x^2 = 0$

$x^2 (x^2 - 2x - 15) = 0$

$x^2 (x - 5) (x + 3) = 0$

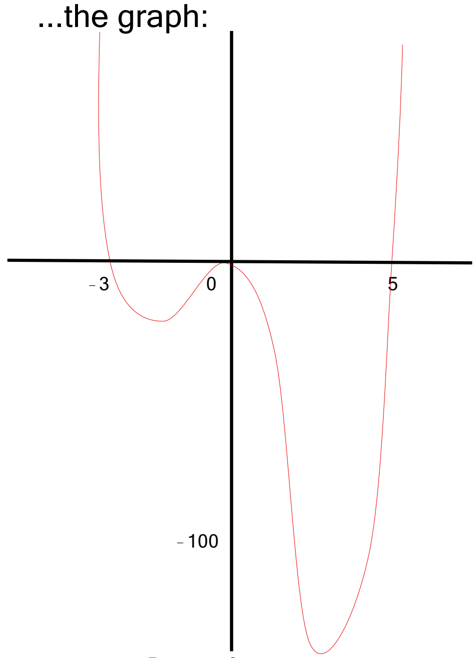
$x = 0$ or 0 or 5 or -3

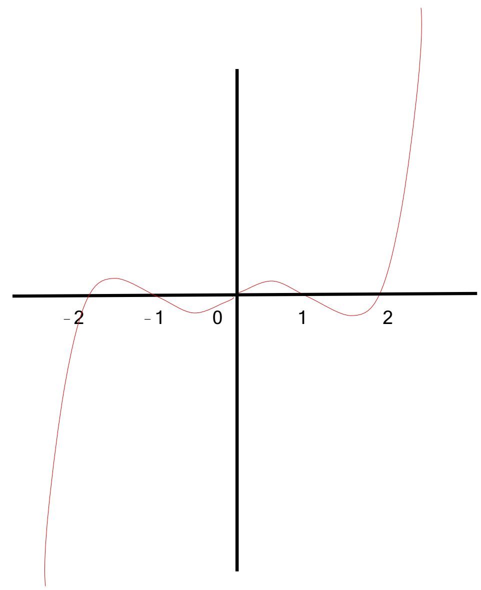
(4) the outer limits: As $x \to\infty$, x^4 dominates the other terms, so $f(x)\to\infty$.

(i.e. f(x) grows unboundedly large)

Likewise, as $x \to\infty$, x^4, dominating the other terms, takes $f(x)\to\infty$

(5) ...the graph:





b. $P(x) = x^5 - 5x^3 + 4x$.

(1) Domain = ℝ

(2) y-intercept: $P(0) = 0$

(3) x-intercepts : $x^5 - 5x^3 + 4x = 0$

when $x(x^4 - 5x^2 + 4) = 0$

$x(x^2 - 4)(x^2 - 1) = 0$

$x(x+2)(x-2)(x+1)(x-1) = 0$

$x = 0, -2, 2, -1, 1$

(4) outer limits: as $x\to\infty$, $f(x)$, dominated by x^5, $\to\infty$

ax $x\to-\infty$, $f(x)$, again ruled by x^5, $\to-\infty$

(5) see above, right. P is an odd function.

Note the curve is smoother than this pathetic illustration indicates.
It is just difficult to draw these curves with a mouse and flaky software.
After the seventeenth try, patience vanishes.

2. Use polynomial long division.

EG: Divide ... $\dfrac{6x^4 + 2x^2 + 22x}{2x^2 + 5}$

$$
\begin{array}{r|l}
 & 3x^2 \qquad - {}^{13}/_2 \\
2x^2 + 5 & \overline{6x^4 + 0 \ + 2x^2 + 22x \ + \ 0} \\
 & 6x^4 \qquad +15x^2 \\
 & \overline{\qquad\qquad -13x^2} \\
 & \qquad\qquad -13x^2 \qquad - {}^{65}/_2 \\
 & \qquad\qquad \overline{\qquad\qquad 22x \ + {}^{65}/_2}
\end{array}
$$

SO $\dfrac{6x^4 + 2x^2 + 22x}{2x^2 + 5} = 3x^2 - {}^{13}/_2 + \dfrac{22x + {}^{65}/_2}{2x^2 + 5}$

OR $6x^4 + 2x^2 + 22x = (3x^2 - {}^{13}/_2)(2x^2 + 5) + 22x + {}^{65}/_2$

Notice the last statement shows the way to check your division results!!

3. a. The graph shows us there is a repeated zero*, of even order, at $x = 0$, and a simple zero at $x = 3$. It also tells us $P(2)$ must be 8.

The zeroes inform us that $(x-0)(x-0)(x-3)$ must be part of the factorization of P. Therefor, $P(x) = x \bullet x \bullet (x-3) \bullet$(other factors). The degree of P , which is 3, tells us that the "other factors" can be only constant.

Stated briefly, we know:
Roots are 0, 0, and 3 & polynomial has degree 3, so
$P(x) = K\, x^2 (x-3)$

Since $P(2) = 8$:
$8 = P(2) = K \bullet 2^2 (2-3) = -4K$, so K must be -2.

Therefor, $P(x) = -2\, x^2 (x-3)$, or $P(x) = -2x^3 + 6\, x^2$

b. Find a second-degree polynomial with real coefficients with roots -3 and 2, passing through (4,3).

r is a root of a polynomial P if, and only if, $(x - r)$ is a factor of $P(x)$.

Having roots -3 and 2 requires that both $(x - -3)$ and $(x - 2)$ be factors.
Since P must be a 2[nd]-degree polynomial, the only other possible factor is a constant.
So:

$P(x) = A\,(x + 3)(x - 2)$ where A is some currently undetermined constant.

The additional requirement that the curve contain (4,3) determines A:

$P(4) = A\,(4+3)(4 - 2) = A\,(7)(2) = 14A$
But $P(4)$ must be 3... so $14A$ must be 3. therefore $A = 3/14$ and
$P(x) = (3/14)(x + 3)(x - 2) = (3/14)(x^2 + x - 6)$

c. Find a fourth degree polynomial with real coefficients, with roots $2i$ and $1 - i$, and constant term 16.

The complex roots of a polynomial with REAL coefficients must occur in conjugate pairs.

Since we are required to find a polynomial with real coefficients, and since $1 - i$ is a root, its conjugate, $1 + i$ must also be a root.
Likewise, since $2i$ is a root, $-2i$ must also be a root. Therefore:

$$P(x) = A\,(x - (1+i))\,(x - (1-i))\,(x - 2i)\,(x + 2i)$$
$$= A\,(x^2 - (1+i+1-i)\,x + (1+i)(1-i))\,(x^2 - 4i^2)$$
$$= A\,(x^2 - 2x + 2)\,(x^2 + 4)$$

The additional requirement that the constant term be 16 determines A.
A(2)(4) must be 16.... thus A must be 2, and $P(x) = 2\,(x^2 - 2x + 2)\,(x^2 + 4)$
$= 2\,(x^4 - 2x^3 + 6x^2 - 8x + 8)$
$= 2x^4 - 4x^3 + 12x^2 - 16x + 16$

4. Find all the zeros of a polynomial function.

a. For $P(x) = 5x^3 - 22x^2 + 18x - 4$, list all the theoretically possible rational roots of P; and use synthetic division to locate one; then find the remaining roots.

RATIONAL roots of a polynomial with INTEGER coefficients must be of the form p/q where p divides the constant term and q divides the leading coefficient.

So any rational roots of $5x^3 - 22x^2 + 18x - 4$ must be of the form:

$$\frac{\pm 4 \text{ or } \pm 2 \text{ or } \pm 1}{\pm 5 \text{ or } \pm 1} \quad \text{which yields:} \quad \pm 1 \quad \pm 2 \quad \pm 4 \quad \pm\frac{1}{5} \quad \pm\frac{2}{5} \quad \pm\frac{4}{5}$$

TESTING we divide $5x^3 - 22x^2 + 18x - 4$ by $x - r$ for each potential root r:

1	5	−22	18	−4	
		5	−17	1	
	5	−17	1	−3	☹

−1	5	−22	18	−4	
		−5	27		
	5	−27			☹

2	5	−22	18	−4	
		10	−24	−12	
	5	−12	−6		☹

−2	5	−22	18	−4	
		−10	−17	1	
	5	−32	1	−3	☹

4	5	−22	18	−4	
		20	−8	40	
	5	−2	10		☹

−4	5	−22	18	−4	
		−10	128	−12	
	5	−32			☹

$^1/_5$	5	−22	18	−4	
		1	$-^{21}/_5$	$^{69}/_{25}$	
	5	−21	$^{69}/_5$		☹

$-^1/_5$	5	−22	18	−4	
		−1	$-^{23}/_5$	☹	
	5	−23	$^{67}/_5$		☹

$^2/_5$	5	−22	18	−4
		2	−8	4
	5	−20	10	0

finally !**

So $P(x) = (x - {}^2/_5)(5x^2 - 20x + 10) = (5x - 2)(x^2 - 4x + 2)$

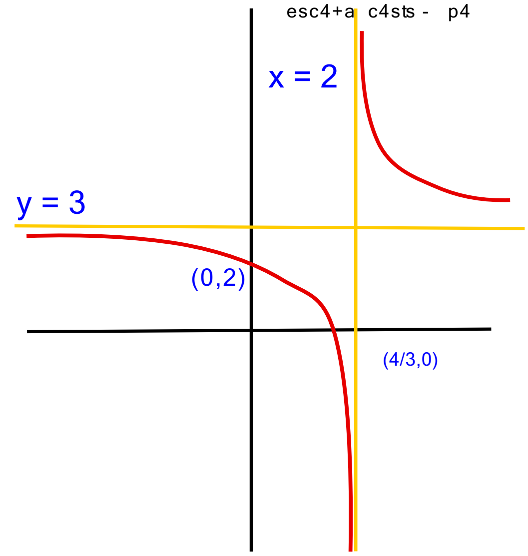
For the remaining roots we solve: $x^2 - 4x + 2 = 0$
... using the quadratic formula: $x = \dfrac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

PS $P(-x) = -5x^3 - 22x^2 - 18x - 4$, so, by Descartes' Rule of Signs, there are no negative real roots.
(We could have saved time **here!!!)

b. Find all the roots of $P(x) = 4x^5 + 15x^3 - 4x$

$4x^5 + 15x^3 - 4x =$ Notice you can factor this!
$x(4x^4 + 15x^2 - 4) =$ ...and this!
$x\,(4x^2 - 1)(x^2 + 4) =$
$x(2x+1)(2x-1)(x + 2i)(x - 2i)$ So the roots of P are: 0, −½ , ½ , −2i , 2i

5. a. $f(x) = \dfrac{3x - 4}{x - 2}$



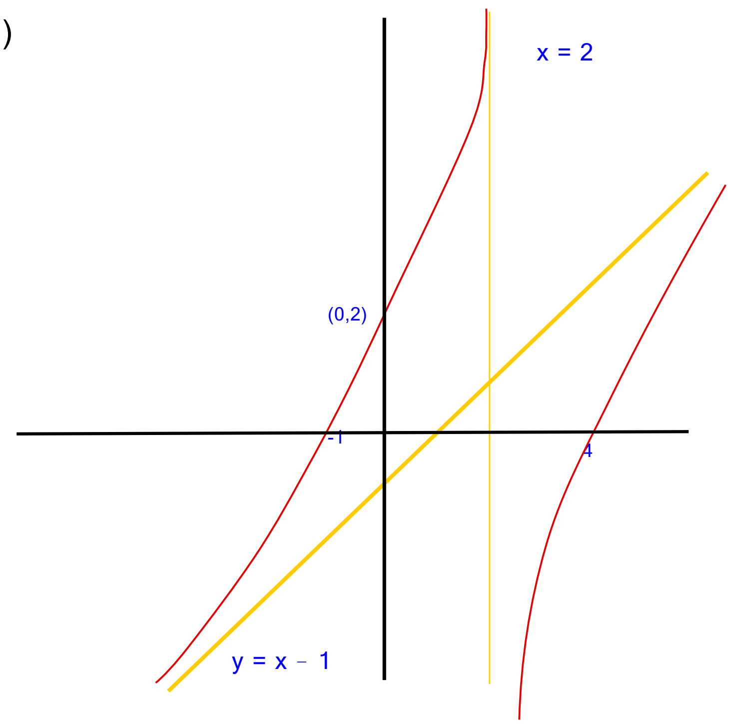
(curve should have no "squiggles".)

- Find the x and y intercepts.
  $f(0) = 2$; $f(x) = 0$ when $3x - 4 = 0$ ... $x = {}^4/_3$
- Find the equation of the vertical asymptote.
  Near x = 2, numerator →2, denominator→0,
  so fraction → ±∞ ... vertical asymptote at x = 2.
- Find the equation of the horizontal asymptote.
  As $x \to \infty$, $(3x - 4)/(x - 2) \to 3$
  so y = 3 is the horizontal asymptote.
- Division also shows us:

$\dfrac{3x - 4}{x - 2} = 3 + \dfrac{2}{x - 2}$, which would have to be ...

y=1/x, shifted 2 units right, stretched vertically by factor 2,
then shifted upwards 3 units.

b. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2} = \dfrac{(x - 4)(x + 1)}{x - 2}$

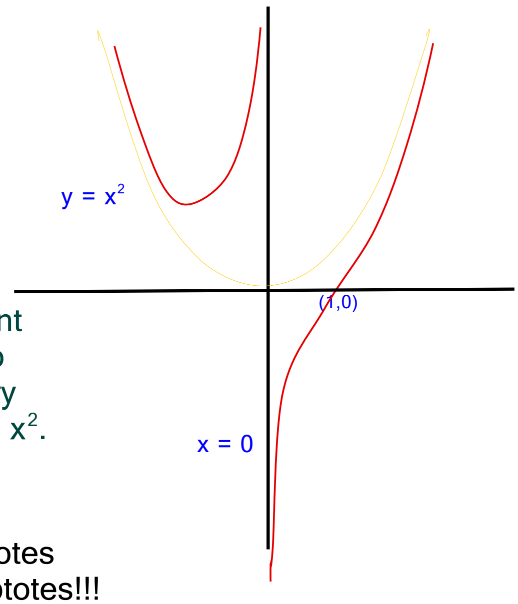


- Intercepts: $f(0) = 2$.
  $f(x) = 0$ when $x = 4, -1$.
- Vertical asymptote
  At x = 2, for the usual reason.
- Horizontal asymptote
  None.
  Divide to see other asymptote:
  $\dfrac{x^2 - 3x - 4}{x - 2} = x - 1 + \dfrac{-6}{x - 2}$
  So we can see that $y = x - 1$ is an oblique linear asymptote for $y = f(x)$.

c. $f(x) = \dfrac{x^3 - 1}{x} = \dfrac{(x-1)(x^2+x+1)}{x}$

x-intercept (1,0)**
vertical asymptote at x = 0

$f(x) = \dfrac{x^3 - 1}{x} = x^2 - \dfrac{1}{x}$



... this shows the given f is x^2 + an amount that is very small when x is very large. So when x is large, the graph of y = f(x) is very close to the graph of the familiar curve $y = x^2$. And as $x \to \infty$, the difference → zero. This is asymptotic behavior.

NO worries: On the test, the only asymptotes you will be asked to find are linear asymptotes!!!

**of course, we determined that the remaining roots are complex:

$x^2 + x + 1 = 0$

$x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}\,i}{2}$

6. The Remainder Theorem gives us the answers:
If polynomial $P(x) = (x - c)\,Q(x) + R(x)$ where Q & R are the quotient and remainder polynomials for $P \div (x - c)$, then R(x) is a constant, and P(c) = that constant.

a. Find the value of $P(x) = 2x^5 - 20x^4 - 20x^3 - 20x^2 - 20x - 22$ @ 11
– without raising 11 to a power.

11	2	–20	–20	–20	–20	–22
		22	22	22	22	22
	2	2	2	2	2	0

This tells us that P(11) = 0, thus 11 is a root, and (x – 11) is a factor of P.
In fact, this tells us: $P(x) = (x - 11)\ 2\ (x^4 + x^3 + x^4 + x^3 + x + 1)$

b. If $P(x) = (x-2)(x^3 - 4x^2 + 7x + 13) + 7$, what is P(2)?

P(2) = (2 – 2) (whatever) + 7

= 0 + 7

SO: P(2) = 7 ... JUST as the remainder theorem says.