The unit distance problem for centrally symmetric convex polygons

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Abstract

Let $f(n)$ be the maximum number of unit distances determined by the vertices of a convex n-gon. Erdős and Moser conjectured that this function is linear. Supporting this conjecture we prove that $f^{sym}(n) \sim 2n$ where $f^{sym}(n)$ is the restriction of $f(n)$ to centrally symmetric convex n-gons. We also present two applications of this result. Given a strictly convex domain K with smooth boundary, if $f_K(n)$ denotes the maximum number of unit segments spanned by n points in the boundary of K, then $f_K(n) = O(n)$ whenever K is centrally symmetric or has width > 1.

1 Introduction

For every finite set of points P in the plane, $f(P)$ denotes the number of unit segments with endpoints in P. We say that P is in *convex position* if P is the vertex set of a strictly convex polygon (no three points are on a line).

More than forty years ago, Erdős and Moser ([7], see also [8], [5], and [10]) initiated the study of the function

$$f(n) = \max\{f(P) : |P| = n, P \text{ in convex position}\}.$$

They proved with a construction that $f(n) \geq \lfloor 5/3(n-1) \rfloor$ and conjectured that $f(n)$ was linearly bounded above. The best known upper bound, $f(n) \leq O(n \log n)$, was first proved by Füredi [9], and very recently by Brass et al. [2], and Brass and Pach [3] using different techniques. The lower bound was improved to $2n-7$ by Edelsbrunner and Hajnal [4], and motivated by this construction, Erdős and Fishburn [6] conjectured that $f(n) < 2n$.

Our main objective is to prove that $f(n)$ restricted to centrally symmetric sets is asymptotically $2n$. This supports both the Erdős-Moser and the Erdős-Fishburn conjectures. In fact, we prove that the function

$$f^{sym}(n) = \max\{f(P) : |P| = n, P \text{ in convex position and centrally symmetric}\}$$

(which only makes sense for even values of n) satisfies

Theorem 1 *For every even $n \geq 2$*

$$2n - \Theta\left(\sqrt{n}\right) \leq f^{sym}(n) \leq 2n - 3.$$

None of the two constructions mentioned above giving lower bounds for $f(n)$, can be extended to a centrally symmetric set. Actually the natural example consisting of the symmetrization of rotated copies of a regular triangle sharing a vertex, only gives $(3/2)\,n \leq f^{sym}(n)$. Even with this in mind, and due in part to the proof of Theorem 1, we conjecture that $f^{sym}(n) \geq 2n - O(1)$.

The proof of the upper bound in Theorem 1 can be extended to a more general result stated below as Theorem 2. We first need to define a family of functions. Let K denote a strictly convex domain in the plane (i.e., a bounded subset of the plane such that if x and y are boundary points of K then the open segment xy is contained in the interior of K) with smooth boundary ∂K. Define

$$f_K(n) = \max\{f(P) : |P| = n \text{ and } P \subset \partial K\},$$

i.e., $f_K(n)$ is the maximum number of unit distances determined by n boundary points of K.

Theorem 2 *If K is centrally symmetric then $f_K(n) \leq 2n - 3$ for every $n \geq 2$.*

We go one step further in this direction by considering a different family of sets K. The following is also an application of Theorem 1.

Theorem 3 *If K has width greater than one then there is $c > 0$ such that $f_K(n) \leq cn$.*

It may be possible to prove the Erdős-Moser conjecture by showing $f_K(n) \leq cn$ for a large class of convex sets K and a universal constant c. Unfortunately, for these purposes, in Theorem 3 $c \to \infty$ when the width approaches one.

From now on given any points x, y in the plane, $H^+(x, y)$ and $H^-(x, y)$ will denote the upper and lower *half-planes* determined by the oriented line $\overrightarrow{xy}$ (we include the line xy in both halfplanes).

2 Proofs of the theorems

Proof of Theorem 1. We first prove the upper bound. Let P denote a convex centrally symmetric polygon with n vertices, and let pq be a diameter of P. Note that if $\|p - q\| < 1$ then $f(P) = 0$. We claim that

$$\text{if } \|p - q\| \geq 1 \text{ then } f(P) \leq 4 + f(P \backslash \{p, q\}) \tag{1}$$

which by induction implies $f(P) \leq 4 + 2(n-2) - 3 = 2n - 3$.

To verify (1) it is enough to show that at most one point in $P \cap H^+(p, q)$ is at distance one from p. Assume that the origin o is the center of symmetry of P. Observe that $q = -p$, otherwise one of the diagonals of the parallelogram $pq(-p)(-q)$ would be longer than the diameter pq. Moreover, since P is centrally symmetric, P must be contained in the closed disk D determined by the circle through p and q centered at o. Let C be the unit circle with center at p, and $u = C \cap pq$. Note that if $p_1, p_2 \in P \cap C \cap H^+(p, q)$ and $\|p_1 - u\| > \|p_2 - u\|$ then $\measuredangle pp_1p_2 < \pi/2$ since $\triangle pp_1p_2$ is isosceles, but $\measuredangle pp_1q \geq \pi/2$ since $p_1 \in D$. Therefore p_2 would be in the interior of $\triangle pqp_1$ contradicting the convexity of P (when $\|p - q\| = 1$ the only possibility is $p_1 = p_2 = q$).

To prove the lower bound we construct a centrally symmetric convex polygon P with $n = k^2 + k$ vertices and at least $2n - 3k$ unit distances among them.

We start with k points in a circle of radius $1/2$. Even though we look at these points as vectors, for simplicity we write their polar coordinates to describe them. Given a fixed $\theta \in (0, \pi)$ let

$$p_j = \left(\frac{1}{2}, \theta_j\right) \text{ where } \theta_1 = 0 \text{ and } \theta_j = 7^{j-k}\theta \text{ for } 2 \leq j \leq k.$$

Now, for every pair (i,j) with $1 \le i < j \le k$, there is a unique point $p_{i,j}$ in $H^+(-p_1,p_1)$ obtained as the intersection of the unit circles with centers $-p_i$ and $-p_j$ (see Figure 1a). Suppose that $(r_{i,j}, \theta_{i,j})$ are the polar coordinates of $p_{i,j}$. By construction we have that $\theta_{i,j} = (\theta_i + \theta_j)/2$, and after some direct calculations

$$r_{i,j} = \frac{1}{2}\left(\sqrt{3+\cos^2\left(\left(\theta_j-\theta_i\right)/2\right)} - \cos\left(\left(\theta_j-\theta_i\right)/2\right)\right).$$

Let $P = \bigcup_{1\le i<j\le k}\{p_{i,j}, -p_{i,j}\} \cup \bigcup_{1\le i\le k}\{p_i, -p_i\}$. Clearly P is centrally symmetric and $|P| = 2\left(\binom{k}{2}+k\right) = n$. Also, each point $p_{i,j}$ is at distance one from $-p_i$ and $-p_j$ which together with the symmetric analogues gives $4\binom{k}{2}$ unit distances. If we add the k unit distances given by the pairs $(p_i,-p_i)$, we get $f(P) \ge 4\binom{k}{2}+k = 2n-3k \ge 2n-3\sqrt{n}$. Finally we argue that P is in convex position.

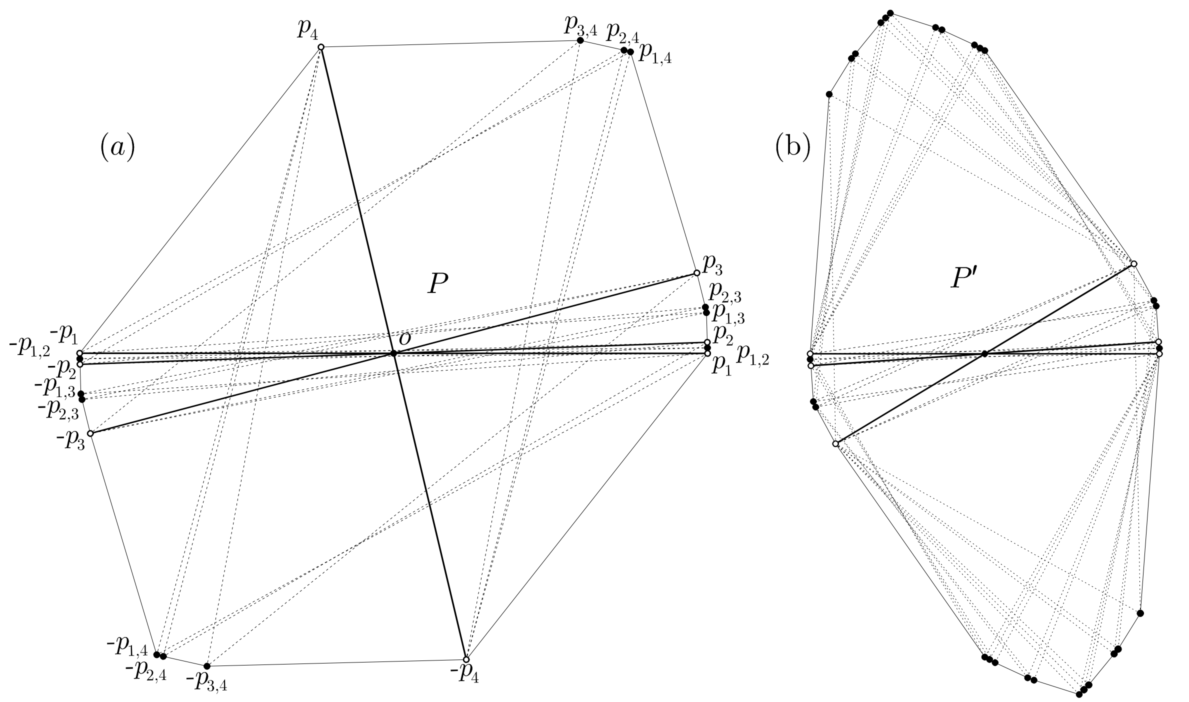

Figure 1: Constructions for the lower bound of Theorem 1.

According to the angles $\theta_{i,j}$ we know that the points

$$p_1, p_{1,2}, p_2, p_{1,3}, p_{2,3}, p_3, \ldots, p_{k-1}, p_{1,k}, p_{2,k}, p_{3,k}, \ldots, p_{k-1,k}, p_k, -p_1$$

are in $H^+(-p_1,p_1)$, and they appear in this order. So by symmetry we just need to show that these $n/2+1$ points are in convex position and $\measuredangle p_{1,2}p_1 o, \measuredangle o(-p_1)p_k < \pi/2$. Observe that the points $p_{1,j}, p_{2,j}, \ldots, p_{j-1,j}, p_j$ are contained in an arc of circle with center at $-p_j$, and thus they are in convex position. Also $\measuredangle o(-p_1)p_k = \measuredangle(-p_1)p_k o < \pi/2$ and for all $2 \le j \le k$, $\measuredangle o p_j p_{j-1,j} = \measuredangle(-p_j)p_j p_{j-1,j} < \pi/2$, $\measuredangle p_{2,j}p_{1,j}o < \measuredangle p_{2,j}p_{1,j}(-p_j) < \pi/2$ (here $p_{2,2} = p_2$).

So it is enough to prove that $\measuredangle o p_{1,j} p_{j-1} < \measuredangle p_{1,j} p_{j-1} o < \pi/2$ for $2 \le j \le k$. The first inequality is given by $\|p_{1,j}\| > \|p_{j-1}\|$, and the second is equivalent to showing that $\langle -p_{j-1}, p_{1,j} - p_{j-1}\rangle > 0$, where $\langle \_, \_ \rangle$ denotes the standard inner product. When $j = 2$ we have $\measuredangle p_{1,2} p_1 o = \measuredangle p_{1,2} p_1 (-p_1) < \pi/2$, and for $j \geq 3$

$$\begin{aligned}\langle -p_{j-1}, p_{1,j} - p_{j-1} \rangle &= \langle p_{j-1}, p_{j-1} \rangle - \langle p_{j-1}, p_{1,j} \rangle \\ &= \|p_{j-1}\|^2 - \|p_{j-1}\| \, \|p_{1,j}\| \cos(\theta_{1,j} - \theta_{j-1}) \\ &= \tfrac{1}{4} - \tfrac{1}{2} r_{1,j} \cos(\theta_j/2 - \theta_{j-1}) \\ &= \tfrac{1}{4}\left(1 - \cos(\theta_j/2 - \theta_{j-1})\left(\sqrt{3 + \cos^2(\theta_j/2)} - \cos(\theta_j/2)\right)\right),\end{aligned}$$

by construction $\theta_j = 7\theta_{j-1}$, thus

$$\langle -p_{j-1}, p_{1,j} - p_{j-1} \rangle = \tfrac{1}{4}\left(1 - \cos\left(\tfrac{5}{2}\theta_{j-1}\right)\left(\sqrt{3 + \cos^2\left(\tfrac{7}{2}\theta_{j-1}\right)} - \cos\left(\tfrac{7}{2}\theta_{j-1}\right)\right)\right).$$

To complete the proof note that the function $g(x) = 1 - \cos(2.5x)\left(\sqrt{3 + \cos^2(3.5x)} - \cos(3.5x)\right)$ is positive in the interval $(0, \pi/7)$ and $\theta_{j-1} \leq \theta_{k-1} = \theta/7 < \pi/7$. □

Remark. We can reduce the error for the lower bound by adding some points to our original construction (see Figure 1b). Given $p \in P$ let $C(p)$ be the unit circle centered at p and

$$h(p) = \begin{cases} C(p) \cap C(-p_1) \cap H^+(-p_1, p_1) & \text{if } p \in H^+(-p_1, p_1) \setminus \{-p_1\} \\ C(p) \cap C(p_k) \cap H^+(-p_1, p_1) & \text{otherwise.} \end{cases}$$

Let $P' = P \cup \bigcup_{p \in P} \{-h(p), h(p)\}$. For θ small enough, it can be verified that P' is in convex position. Also $|P'| = 3|P| - 2 = 3k^2 + 3k - 2 = n'$ and $f(P') \geq f(P) + 2(2|P| - 2) \geq 2n' - 3k \geq 2n' - \sqrt{3n'}$.

Finally, by deleting an appropriate number of points from P', one can show that for an arbitrary even $n \geq 2$, $f^{sym}(n) \geq 2n - 3 - \sqrt{3n}$.

Proof of Theorem 2. Consider any n-point subset P of ∂K. Let P' be the symmetric of P in ∂K. The previous proof guarantees that each of the endpoints of the diameter of $P \cup P'$ is at distance one of at most two other elements in $P \cup P'$. Moreover, one of these points is in P. The rest follows by induction. □

Proof of Theorem 3. The directed closed segment xy is a *chord of K in direction α* if $x, y \in \partial K$ and the argument of the vector $y - x$ is α. For each $\alpha \in [0, 2\pi)$ we say that xy is the *α-directional diameter of K*, or simply the *α-diameter*, if xy is the longest chord of K in direction α (this is well defined because K is strictly convex). We also denote by a_α, b_α the endpoints of the unique unit chord of K in direction α with the property that any chord parallel to $a_\alpha b_\alpha$ contained in $H^-(a_\alpha, b_\alpha)$ has length less than one. We call $a_\alpha b_\alpha$ the *α-unit chord of K*. We need the next lemma for the proof of the theorem.

Lemma 1 *Any two directional diameters of K intersect in their interior.*

Proof. Let xy and zw be the α- and β-diameters of K and suppose that they do not intersect. Then the quadrilateral with vertices x, y, z, w is convex and xy, zw are opposite sides (see Figure 2). Assume that $xyzw$ is the order of the vertices in the quadrilateral. Since the internal angles add up to 2π then we can assume that $\measuredangle yxw + \measuredangle zyx \geq \pi$. Since K is strictly convex then there is a chord parallel to xy in $H^+(x, y)$ with length greater than xy which contradicts the fact that xy

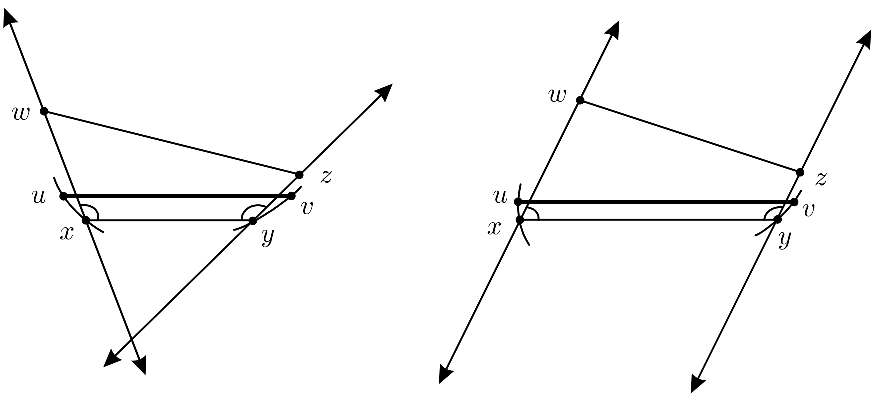

Figure 2: The chord uv is larger than the chord xy.

is the α-diameter. Finally note that even if $x = w$ (or $z = y$), we can replace the line xw (or zy) by the tangent line to K at x (or at y) and the argument still follows (here we use the smoothness assumption). □

The last lemma, together with the continuity of ∂K, guarantees that for any boundary point x of K there exists a unique directional diameter with x as one of its endpoints. It also shows that both the left and right endpoints of the α-diameter (as functions of α) move continuously counterclockwise in ∂K. For each α-unit chord look at the two directional diameters having a_α or b_α as one of their endpoints, and let c_α be their point of intersection. Let $\theta(\alpha) = \pi - \angle b_\alpha c_\alpha a_\alpha$ (see Figure 3). Since K has width greater than one then θ is a strictly positive function. Hence,

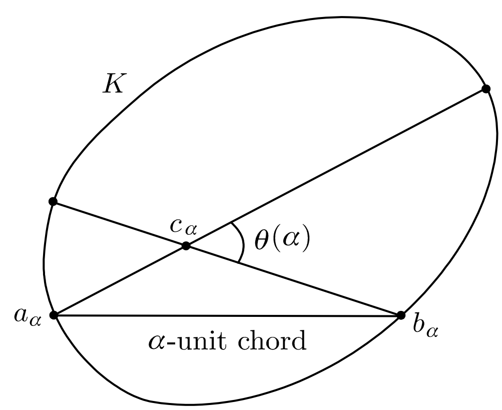

Figure 3: Definition of $\theta(\alpha)$.

by continuity of θ on the compact set $[0, 2\pi]$ we have $m = \min\{\theta(\alpha) : \alpha \in [0, 2\pi]\} > 0$.

Let P be an n-point subset of ∂K. Define

$$U = \{\alpha \in [0, 2\pi) : a_\alpha, b_\alpha \in P\}$$

and for every $\beta \in [0, \pi)$

$$N(\beta) = |\{\alpha \in U : \text{the } \alpha\text{-unit chord does not cross the } \beta\text{-diameter}\}|.$$

Observe that

$$N(\beta) = \sum_{\alpha \in U} \chi_\beta(\alpha)$$

where

$$\chi_\beta(\alpha) = \begin{cases} 1 \text{ if the } \alpha\text{-unit chord does not intersect the } \beta\text{-diameter} \\ 0 \text{ otherwise.} \end{cases}$$

So

$$\int_0^\pi N(\beta)\, d\beta = \int_0^\pi \sum_{\alpha \in U} \chi_\beta(\alpha)\, d\beta = \sum_{\alpha \in U} \int_0^\pi \chi_\beta(\alpha)\, d\beta,$$

but for fixed α, if b_α is an endpoint of the δ-diameter then $\chi_\beta(\alpha) = 1$ if and only if $\beta \in [\delta, \delta + \theta(\alpha)]$, i.e., $\int_0^\pi \chi_\beta(\alpha)\, d\beta = \theta(\alpha)$. Therefore

$$mf(P) = m|U| \leq \sum_{\alpha \in U} \theta(\alpha) = \int_0^\pi N(\beta)\, d\beta.$$

Now, as an application of Theorem 1 we claim that

$$N(\beta) < 2n \text{ for all } \beta \in [0, \pi) \tag{2}$$

and so

$$\int_0^\pi N(\beta)\, d\beta \leq 2\pi n.$$

Thus $f(P) \leq \left(\frac{2\pi}{m}\right) n$ for all n-point subsets of ∂K. Hence $f_K(n) \leq \left(\frac{2\pi}{m}\right) n$.

To prove (2) let xy be the β-diameter. First suppose that $|\{x,y\} \cap P| \leq 1$, let $P_1 = P \cap H^+(x,y)$, $P_2 = P \cap H^-(x,y)$, and P_1', P_2' be the sets obtained from P_1 and P_2 by symmetrization with respect to the midpoint of xy. Since xy is a directional diameter then the sets $P_1 \cup P_1'$ and $P_2 \cup P_2'$ are in convex position, so according to Theorem 1, for $i = 1, 2$

$$2f(P_i) = f(P_i) + f(P_i') \leq f\left(P_i \cup P_i'\right) \leq 2\left|P_i \cup P_i'\right| - 3 = 4|P_i| - 3,$$

therefore

$$N(\beta) = f(P_1) + f(P_2) \leq 2(|P_1| + |P_2|) - 3 \leq 2n - 1 < 2n.$$

If $x, y \in P$ then $|P_i \cup P_i'| = 2|P_i| - 2$ in the above analysis, so even though $|P_1| + |P_2| = n + 2$ the conclusion still holds. □

Corollary 1 *Let K be a strictly convex domain with C^2 boundary. If the curvature of K is less than 2 at each point of ∂K then $f_K(n) \leq cn$ for some positive constant c that only depends on K.*

Proof. By Blaschke's Rolling Theorem [1] if the curvature of K is less than 2 at each point of ∂K then a circle of radius $1/2$ can freely roll inside K, and therefore the width of K is greater than one. □

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