On the maximum number of isosceles right triangles in a finite point set

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Abstract

Let Q be a finite set of points in the plane. For any set P of points in the plane, $S_Q(P)$ denotes the number of similar copies of Q contained in P. For a fixed n, Erdős and Purdy asked to determine the maximum possible value of $S_Q(P)$, denoted by $S_Q(n)$, over all sets P of n points in the plane. We consider this problem when $Q = \triangle$ is the set of vertices of an isosceles right triangle. We give exact solutions when $n \leq 9$, and provide new upper and lower bounds for $S_\triangle(n)$.

1 Introduction

In the 1970s Paul Erdős and George Purdy [6, 7, 8] posed the question, "Given a finite set of points Q, what is the maximum number of similar copies $S_Q(n)$ that can be determined by n points in the plane?". This problem remains open in general. However, there has been some progress regarding the order of magnitude of this maximum as a function of n. Elekes and Erdős [5] noted that $S_Q(n) \leq n(n-1)$ for any pattern Q and they also gave a quadratic lower bound for $S_Q(n)$ when $|Q| = 3$ or when all the coordinates of the points in Q are algebraic numbers. They also proved a slightly subquadratic lower bound for all other patterns Q. Later, Laczkovich and Ruzsa [9] characterized precisely those patterns Q for which $S_Q(n) = \Theta(n^2)$. In spite of this, the coefficient of the quadratic term is not known for any non-trivial pattern; it is not even known if $\lim_{n\to\infty} S_Q(n)/n^2$ exists!

Apart from being a natural question in Discrete Geometry, this problem also arose in connection to optimization of algorithms designed to look for patterns among data obtained from scanners, digital cameras, telescopes, etc. (See [2, 3, 4] for further references.)

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Our paper considers the case when Q is the set of vertices of an isosceles right triangle. The case when Q is the set of vertices of an equilateral triangle has been considered in [1]. To avoid redundancy, we refer to an isosceles right triangle as an IRT for the remainder of the paper. We begin with some definitions. Let P denote a finite set of points in the plane. We define $S_\triangle(P)$ to be the number of triplets in P that are the vertices of an IRT. Furthermore, let

$$S_\triangle(n) = \max_{|P|=n} S_\triangle(P).$$

As it was mentioned before, Elekes and Erdős established that $S_\triangle(n) = \Theta(n^2)$ and it is implicit from their work that $1/18 \leq \liminf_{n\to\infty} S_\triangle(n)/n^2 \leq 1$. The main goal of this paper is to derive improved constants that bound the function $S_\triangle(n)/n^2$. Specifically, in Sections 2 and 3, we prove the following result:

Theorem 1.

$$0.433064 < \liminf_{n\to\infty} \frac{S_\triangle(n)}{n^2} \leq \frac{2}{3} < 0.66667.$$

We then proceed to determine, in Section 4, the exact values of $S_\triangle(n)$ when $3 \leq n \leq 9$. Several ideas for the proofs of these bounds come from the equivalent bounds for equilateral triangles in [1].

2 Lower Bound

We use the following definition. For $z \in P$, let $R_{\pi/2}(z, P)$ be the $\pi/2$ counterclockwise rotation of P with center z. Furthermore, let $\deg_{\pi/2}(z)$ be the number of isosceles right triangles in P such that z is the right-angle vertex of the triangle. If $z \in P$, then $\deg_{\pi/2}(z)$ can be computed by simply rotating our point set P by $\pi/2$ about z and counting the number of points in the intersection other than z. Therefore,

$$\deg_{\pi/2}(z) = |P \cap R_{\pi/2}(z, P)| - 1. \tag{1}$$

Due to the fact that an IRT has only one right angle, then

$$S_\triangle(P) = \sum_{z\in P} \deg_{\pi/2}(z).$$

That is, the sum computes the number of IRTs in P. From this identity an initial $5/12$ lower bound can be derived for $\liminf_{n\to\infty} S_\triangle(n)/n^2$ using the set

$$P = \left\{(x, y) \in \mathbb{Z}^2 : 0 \leq x \leq \sqrt{n}, 0 \leq y \leq \sqrt{n}\right\}.$$

We now improve this bound.

The following theorem generalizes our method for finding a lower bound. We denote by Λ the lattice generated by the points $(1, 0)$ and $(0, 1)$; furthermore, we refer to points in Λ as *lattice points*. The next result provides a formula for the leading term of $S_\triangle(P)$ when our points in P are lattice points enclosed by a given shape. This theorem, its proof, and notation, are similar to Theorem 2 in [1], where the authors obtained a similar result for equilateral triangles in place of IRTs.

Theorem 2. *Let K be a compact set with finite perimeter and area 1. Define $f_K : \mathbb{C} \to \mathbb{R}^+$ as $f_K(z) = Area(K \cap R_{\pi/2}(z, K))$ where $z \in K$. If K_n is a similar copy of K intersecting Λ in exactly n points, then*

$$S_\triangle(K_n \cap \Lambda) = \left(\int_K f_K(z)\, dz \right) n^2 + O(n^{3/2}).$$

Proof. Given a compact set L with finite area and perimeter, we have that

$$|rL \cap \Lambda| = \text{Area}(rL) + O(r) = r^2 \text{Area}(L) + O(r),$$

where rL is the scaling of L by a factor r. Therefore,

$$\begin{aligned} S_\triangle(K_n \cap \Lambda) &= \sum_{z \in K_n \cap \Lambda} |(\Lambda \cap K_n) \cap R_{\pi/2}(z, (K_n \cap \Lambda))| - 1 \\ &= \sum_{z \in K_n \cap \Lambda} \text{Area}(K_n \cap R_{\pi/2}(z, K_n)) + O(\sqrt{n}). \end{aligned}$$

We see that each error term in the sum is bounded by the perimeter of K_n, which is finite by hypothesis. Thus,

$$\begin{aligned} S_\triangle(K_n \cap \Lambda) &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n^2} \text{Area}(K_n \cap R_{\pi/2}(z, K_n)) + O(n^{3/2}) \\ &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n} \text{Area}(\frac{1}{\sqrt{n}} \left(K_n \cap R_{\pi/2}(z, K_n)\right)) + O(n^{3/2}) \\ &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n} \text{Area} \left(\frac{1}{\sqrt{n}} K_n \cap R_{\pi/2} \left(\frac{z}{\sqrt{n}}, \frac{1}{\sqrt{n}} K_n \right) \right) + O(n^{3/2}). \end{aligned}$$

The last sum is a Riemann approximation for the function $f_{(1/\sqrt{n})K_n}$ over the region $(1/\sqrt{n})K_n$, thus

$$S_\triangle(K_n \cap \Lambda) = n^2 \left(\int_{\frac{1}{\sqrt{n}} K_n} f_{\frac{1}{\sqrt{n}} K_n}(z)\, dz + O\left(\frac{1}{\sqrt{n}} \right) \right) + O(n^{3/2}).$$

Since

$$\text{Area}\left(\frac{1}{\sqrt{n}} K_n \right) = \frac{1}{n} \text{Area}(K_n) = \frac{1}{n}(n + O(\sqrt{n})) = 1 + O\left(\frac{1}{\sqrt{n}} \right) = \text{Area}(K) + O\left(\frac{1}{\sqrt{n}} \right),$$

it follows that,

$$\int_{\frac{1}{\sqrt{n}} K_n} f_{\frac{1}{\sqrt{n}} K_n}(z)\, dz = \int_K f_K(z)\, dz + O\left(\frac{1}{\sqrt{n}} \right).$$

As a result,

$$\begin{aligned} S_\triangle(K_n \cap \Lambda) &= n^2 \int_{\frac{1}{\sqrt{n}} K_n} f_{\frac{1}{\sqrt{n}} K_n}(z)\, dz + O(n^{3/2}) \\ &= n^2 \int_K f_K(z)\, dz + O(n^{3/2}). \end{aligned}$$

□

The importance of this theorem can be seen immediately. Although our 5/12 lower bound for $\liminf_{n\to\infty} S_\triangle(n)/n^2$ was derived by summing the degrees of each point in a square lattice, the same result can be obtained by letting K be the square $\{(x,y) : |x| \le \frac{1}{2}, |y| \le \frac{1}{2}\}$. It follows that $f_K(x,y) = (1-|x|-|y|)(1-||x|-|y||)$ and

$$S_\triangle(K_n \cap \Lambda) = \left(\int_K f_K(z)\,dz\right) n^2 + O(n^{3/2}) = \frac{5}{12}n^2 + O(n^{3/2}).$$

An improved lower bound will follow provided that we find a set K such that the value for the integral in Theorem 2 is larger than 5/12. We get a larger value for the integral by letting K be the circle $\{z \in \mathbb{C} : |z| \le 1/\sqrt{\pi}\}$. In this case

$$f_K(z) = \frac{2}{\pi}\arccos(\frac{\sqrt{2\pi}}{2}|z|) - |z|\sqrt{\frac{2}{\pi} - |z|^2} \tag{2}$$

and

$$S_\triangle(K_n \cap \Lambda) = \left(\int_K f_K(z)\,dz\right) n^2 + O(n^{3/2}) = \left(\frac{3}{4} - \frac{1}{\pi}\right) n^2 + O(n^{3/2}).$$

It was conjectured in [1] that not only does $\lim_{n\to\infty} E(n)/n^2$ exist, but it is attained by the uniform lattice in the shape of a circle. ($E(n)$ denotes the maximum number of equilateral triangles determined by n points in the plane.) The corresponding conjecture in the case of the isosceles right triangle turns out to be false. That is, if $\lim_{n\to\infty} S_\triangle(n)/n^2$ exists, then it must be strictly greater than $3/4 - 1/\pi$. Define $\overline{\Lambda}$ to be the translation of Λ by the vector $(1/2, 1/2)$. The following lemma will help us to improve our lower bound.

Lemma 1. *If $(j,k) \in \mathbb{R}^2$ and $\Lambda' = \Lambda$ or $\Lambda' = \overline{\Lambda}$, then*

$$R_{\pi/2}((j,k),\Lambda') \cap \Lambda' = \begin{cases} \Lambda' \text{ if } (j,k) \in \Lambda \cup \overline{\Lambda}, \\ \varnothing \text{ else.} \end{cases}$$

Proof. Observe that

$$R_{\pi/2}((j,k),(s,t)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s-j \\ t-k \end{pmatrix} + \begin{pmatrix} j \\ k \end{pmatrix} = \begin{pmatrix} k-t+j \\ s-j+k \end{pmatrix}.$$

First suppose $(s,t) \in \Lambda$. Since $s,t \in \mathbb{Z}$, then $(k-t+j, s-j+k) \in \Lambda$ if and only if $k-j \in \mathbb{Z}$ and $k+j \in \mathbb{Z}$. This can only happen when either both j and k are half-integers (i.e., $(j,k) \in \overline{\Lambda}$), or both j and k are integers (i.e., $(j,k) \in \Lambda$). Now suppose $(s,t) \in \overline{\Lambda}$. In this case, because both s and t are half-integers, we conclude that $(k-t+j, s-j+k) \in \overline{\Lambda}$ if and only if both $k-j \in \mathbb{Z}$ and $k+j \in \mathbb{Z}$. Once again this occurs if and only if $(j,k) \in \Lambda \cup \overline{\Lambda}$. $\square$

Recall that if K denotes the circle of area 1, then $(3/4 - 1/\pi)n^2$ is the leading term of $S_\triangle(K_n \cap \Lambda)$. The previous lemma implies that, if we were to adjoin a point $z \in \mathbb{R}^2$ to $K_n \cap \Lambda$ such that z has half-integer coordinates and is located near the center of the circle formed by the points of $K_n \cap \Lambda$, then $\deg_{\pi/2}(z)$ will approximately equal $|K_n \cap \Lambda|$. We obtain the next theorem by further exploiting this idea.

Theorem 3.

$$.43169 \approx \frac{3}{4} - \frac{1}{\pi} < .433064 < \liminf_{n\to\infty} \frac{S_\triangle(n)}{n^2}$$

Proof. Let K be the circle of area 1, $A = K_{m_1} \cap \Lambda$, and $B = K_{m_2} \cap \overline{\Lambda}$. Moreover, position B so that its points are centered on the circle formed by the points in A (See Figure 1). We let $n = m_1 + m_2 = |A \cup B|$ and $m_2 = x \cdot m_1$, where $0 < x < 1$ is a constant to be determined.

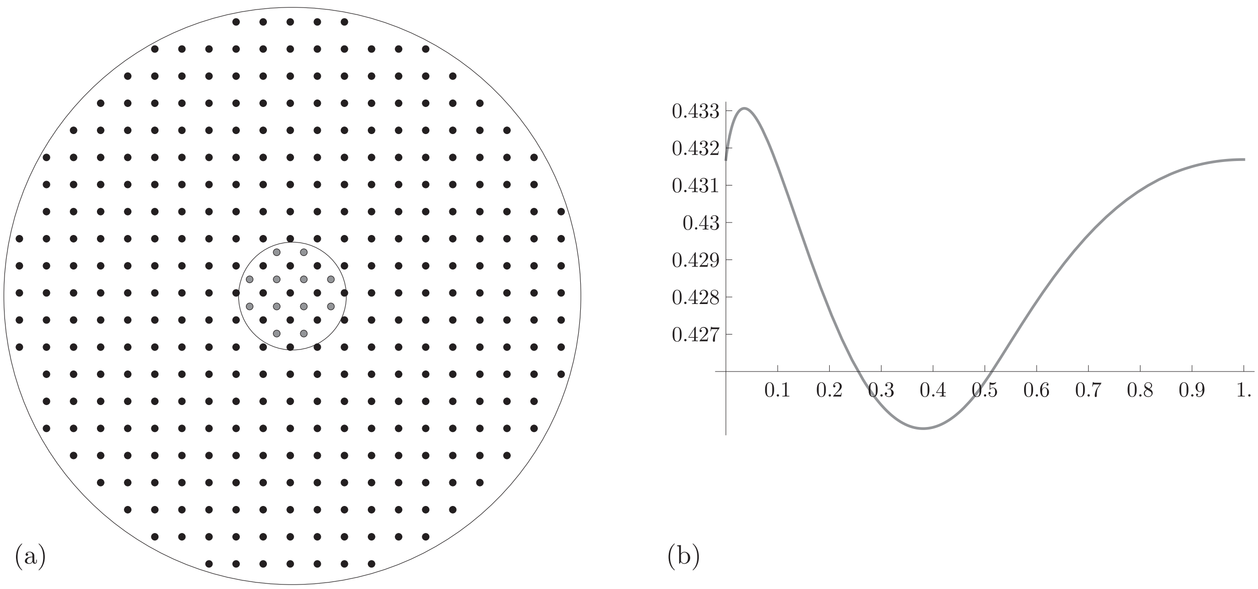


Figure 1: (a) Set B (gray points) centered on set A (black points), (b) Plot of the n^2 coefficient of $S_\triangle(A \cup B)$ as x ranges from 0 to 1.

We proceed to maximize the leading coefficient of $S_\triangle(A \cup B)$ as x varies from 0 to 1. By Lemma 1, there cannot exist an IRT whose right-angle vertex lies in A while one $\pi/4$ vertex lies in A and the other lies in B. Similarly, there cannot exist an IRTwhose right angle-vertex lies in B while one $\pi/4$ vertex lies in A and the other lies in B. Therefore, each IRTwith vertices in $A \cup B$ must fall under one of the following four cases:

Case 1: All three vertices in A. Using Theorem 2, it follows that there are $(3/4 - 1/\pi)m_1^2 + O(m_1^{3/2})$ IRTs in this case. Since $m_1 = n/(1+x)$, the number of IRTs in terms of n equals

$$\left(\frac{3}{4} - \frac{1}{\pi}\right) \frac{n^2}{(1+x)^2} + O(n^{3/2}). \tag{3}$$

Case 2: All three vertices in B. By Theorem 2, there are $(3/4 - 1/\pi)m_2^2 + O(m_2^{3/2})$ IRTs in this case. This time $m_2 = nx/(1+x)$ and the number of IRTs in terms of n equals

$$\left(\frac{3}{4} - \frac{1}{\pi}\right) \frac{n^2 x^2}{(1+x)^2} + O(n^{3/2}). \tag{4}$$

Case 3: Right-angle vertex in B, $\pi/4$ vertices in A. The relationship given by Lemma 1 allows us to slightly adapt the proof of Theorem 2 in order to compute the number of IRTs in this case. The integral approximation to the number of IRTs in this case is given by

$$\sum_{z\in K_{m_2}\cap\overline{\Lambda}} |(K_{m_1}\cap\Lambda)\cap R_{\pi/2}(z,(K_{m_1}\cap\Lambda))| = m_1^2\left(\int_{\frac{1}{\sqrt{m_1}}K_{m_2}} f_{\frac{1}{\sqrt{m_1}}K_{m_1}}(z)\,dz\right) + O(m_1^{3/2}).$$

But

$$\text{Area}\left(\frac{1}{\sqrt{m_1}}K_{m_2}\right) = \text{Area}\left(\sqrt{\frac{m_2}{m_1}}K\right) + O(\sqrt{m_1}),$$

so

$$m_1^2\left(\int_{\frac{1}{\sqrt{m_1}}K_{m_2}} f_{\frac{1}{\sqrt{m_1}}K_{m_1}}(z)\,dz\right) + O(m_1^{3/2}) = m_1^2\left(\int_{\sqrt{\frac{m_2}{m_1}}K} f_K(z)\,dz\right) + O(m_1^{3/2}).$$

Expressing this value in terms of n gives

$$\left(\int_{\sqrt{x}K} f_K(z)\,dz\right)\frac{n^2}{(1+x)^2} + O(n^{3/2}). \tag{5}$$

Case 4: Right-angle vertex in A, $\pi/4$ vertices in B. As in Case 3, the number of IRTs is given by

$$\sum_{z\in K_{m_1}\cap\Lambda} |(K_{m_2}\cap\overline{\Lambda})\cap R_{\pi/2}(z,(K_{m_2}\cap\overline{\Lambda}))| = m_2^2\left(\int_{\frac{1}{\sqrt{m_2}}K_{m_1}} f_{\frac{1}{\sqrt{m_2}}K_{m_2}}(z)\,dz\right) + O(m_2^{3/2}). \tag{6}$$

Now recall that $f_{(1/\sqrt{m_2})K_{m_2}}(z) = \text{Area}\big((1/\sqrt{m_2})K_{m_2}\cap R_{\pi/2}(z,(1/\sqrt{m_2})K_{m_2})\big)$. It follows that $f_{(1/\sqrt{m_2})K_{m_2}}(z_0) = 0$ if and only if z_0 is farther than $\sqrt{2/\pi}$ from the center of $(1/\sqrt{m_2})K_{m_2}$. Thus for small enough values of m_2, the region of integration in Equation (6) is actually $(\sqrt{2/m_2})K_{m_2}$, so it does not depend on m_1. We consider two subcases.

First, if $x\le 1/2$ (i.e., $m_2\le m_1/2$), then

$$\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{m_2}}\frac{\sqrt{2m_2}}{\sqrt{\pi}} \le \frac{1}{\sqrt{m_2}}\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{m_1}}{\sqrt{2}} = \frac{1}{\sqrt{m_2}}\sqrt{\frac{m_1}{\pi}}.$$

The left side of the above inequality is the radius of $(\sqrt{2/m_2})K_{m_2}$, meanwhile the right side is the radius of $(1/\sqrt{m_2})K_{m_1}$, thus the region of integration where $f_{\frac{1}{\sqrt{m_2}}K_{m_2}}$ is nonzero equals $(\sqrt{2/m_2})K_{m_2}$. Hence, the number of IRTs equals

$$m_2^2\left(\int_{\sqrt{\frac{2}{m_2}}K_{m_2}} f_{\frac{1}{\sqrt{m_2}}K_{m_2}}(z)\,dz\right) + O(m_2^{3/2}) = m_2^2\left(\int_{\sqrt{2}K} f_K(z)\,dz\right) + O(m_2^{3/2})$$

$$= \left(\int_{\sqrt{2}K} f_K(z)\,dz\right)x^2n^2 + O(n^{3/2}). \tag{7}$$

Now we consider the case $x > 1/2$ (i.e., $m_2 > m_1/2$). In this case, $f_{\frac{1}{\sqrt{m_2}}K_{m_2}}$ is nonzero for all points in $\frac{1}{\sqrt{m_2}}K_{m_1}$. Thus the number of IRTs in this case equals

$$\begin{aligned} m_2^2\left(\int_{\frac{1}{\sqrt{m_2}}K_{m_1}} f_{\frac{1}{\sqrt{m_2}}K_{m_2}}(z)\,dz\right) + O(m_2^{3/2}) &= m_2^2\left(\int_{\sqrt{\frac{m_1}{m_2}}K} f_K(z)\,dz\right) + O(m_2^{3/2}) \\ &= \left(\int_{\sqrt{\frac{1}{x}}K} f_K(z)\,dz\right)\frac{n^2x^2}{(1+x)^2} + O(n^{3/2}) \qquad (8) \end{aligned}$$

By Equation (2), we have that for $t > 0$,

$$\begin{aligned} \int_{tK} f_K(z)\,dz =& 2\pi\int_0^{t/\sqrt{\pi}}\left(\frac{2}{\pi}\arccos(\frac{\sqrt{2\pi}}{2}r) - r\sqrt{\frac{2}{\pi}-r^2}\right) r\,dr \\ =& \frac{1}{2\pi}\left(4t^2\arccos(\frac{t}{\sqrt{2}}) + 2\arcsin(\frac{t}{\sqrt{2}}) - t(t^2+1)\sqrt{2-t^2}\right). \end{aligned}$$

Therefore, putting all four cases together (i.e., expressions (3), (4), (5), and either (7) or (8)), we obtain that the n^2 coefficient of $S_\triangle(A\cup B)$ equals

$$\frac{1}{4\pi(x+1)^2}\left(8x\arccos\sqrt{\frac{x}{2}} + 4\arcsin\sqrt{\frac{x}{2}} + (5\pi-4)x^2 + (3\pi-4) - 2(x+1)\sqrt{2x-x^2}\right)$$

if $0 < x \le 1/2$, or

$$\begin{aligned} \frac{1}{4\pi(x+1)^2}\Bigg(8x\left(\arccos\sqrt{\frac{x}{2}} + \arccos\sqrt{\frac{1}{2x}}\right) + 4\arcsin\sqrt{\frac{x}{2}} + 4x^2\arcsin\sqrt{\frac{1}{2x}} + \\ (3\pi-4)(x^2+1) - 2(x+1)\left(\sqrt{2x-x^2} + \sqrt{2x-1}\right)\Bigg) \end{aligned}$$

if $1/2 < x < 1$. Letting x vary from 0 to 1, it turns out that this coefficient is maximized (see Figure 1) when $x \approx .0356067$ (this corresponds to when the radius of B is approximately 18.87% of the radius of A). Letting x equal this value gives 0.433064 as a decimal approximation to the maximum value attained by the n^2 coefficient. □

At this point, one might be tempted to further increase the quadratic coefficient by placing a third set of lattice points arranged in a circle and centered on the circle formed by B. It turns out that forming such a configuration does not improve the results in the previous theorem. This is due to Lemma 1. More specifically, given our construction from the previous theorem, there is no place to adjoin a point z to the center of $A\cup B$ such that $z \in \Lambda$ or $z \in \overline{\Lambda}$. Hence, if we were to add the point z to the center of $A\cup B$, then any new IRTs would have their right-angle vertex located at z with one $\pi/4$ vertex in A and the other $\pi/4$ vertex in B. Doing so can produce at most $2m_2 = 2xm_1 \approx .0712m_1$ new IRTs (recall that $x \approx .0356066$ in our construction). On the other hand, adding z to the perimeter of A, gives us $m_1 f_K(1/\sqrt{\pi}) \approx .1817m_1$ new IRTs.

3 Upper Bound

We now turn our attention to finding an upper bound for $S_\triangle(n)/n^2$. It is easy to see that $S_\triangle(n) \leq n^2 - n$, since any pair of points can be the vertices of at most 6 IRTs. Our next theorem improves this bound. The idea is to prove that there exists a point in P that does not belong to many IRTs. First, we need the following definition.

For every $z \in P$, let $R^+_{\pi/4}(z, P)$ and $R^-_{\pi/4}(z, P)$ be the dilations of P, centered at z, by a factor of $\sqrt{2}$ and $1/\sqrt{2}$, respectively; followed by a $\pi/4$ counterclockwise rotation with center z. Furthermore, let $\deg^+_{\pi/4}(z)$ and $\deg^-_{\pi/4}(z)$ be the number of isosceles right triangles zxy with $x, y \in P$ such that zxy is ordered in counterclockwise order, and zy, respectively zx, is the hypotenuse of the triangle zxy.

Much like the case of $\deg_{\pi/2}$, $\deg^+_{\pi/4}$ and $\deg^-_{\pi/4}$ can be computed with the following identities,

$$\deg^+_{\pi/4}(z) = \left| P \cap R^+_{\pi/4}(z, P) \right| - 1 \text{ and } \deg^-_{\pi/4}(z) = \left| P \cap R^-_{\pi/4}(z, P) \right| - 1.$$

Theorem 4. *For $n \geq 3$,*

$$S_\triangle(n) \leq \left\lfloor \frac{2}{3}(n-1)^2 - \frac{5}{3} \right\rfloor.$$

Proof. By induction on n. If $n = 3$, then $S_\triangle(3) \leq 1 = \lfloor (2 \cdot 4 - 5)/3 \rfloor$. Now suppose the theorem holds for $n = k$. We must show this implies the theorem holds for $n = k+1$. Suppose that there is a point $z \in P$ such that $\deg_{\pi/2}(z) + \deg^+_{\pi/4}(z) + \deg^-_{\pi/4}(z) \leq \lfloor (4n-5)/3 \rfloor$. Then by induction,

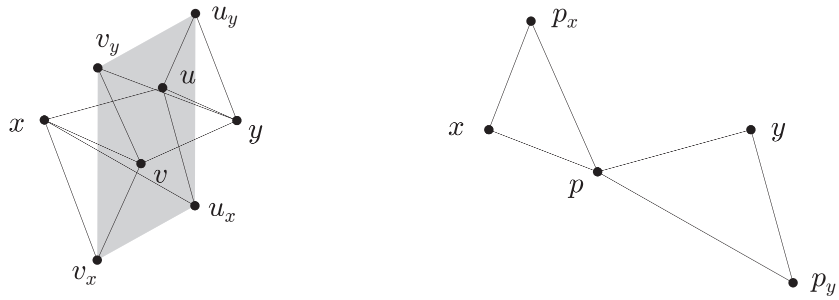
$$\begin{aligned} S_\triangle(k+1) &\leq \deg_{\pi/2}(z) + \deg^+_{\pi/4}(z) + \deg^-_{\pi/4}(z) + S_\triangle(k) \\ &\leq \left\lfloor \frac{4k-1}{3} \right\rfloor + \left\lfloor \frac{2}{3}(k-1)^2 - \frac{5}{3} \right\rfloor = \left\lfloor \frac{2}{3}k^2 - \frac{5}{3} \right\rfloor. \end{aligned}$$

The last equality can be verified by considering the three possible residues of k when divided by 3. Hence, our theorem is proved if we can find a point $z \in P$ with the desired property.

Let $x, y \in P$ be points such that x and y form the diameter of P. In other words, if $w \in P$, then the distance from w to any other point in P is less than or equal to the distance from x to y. We now prove that either x or y is a point with the desired property mentioned above. We begin by analyzing $\deg^-_{\pi/4}$. We use the same notation from Theorem 1 in [1].

Define $N_x = P \cap R^-_{\pi/4}(x, P) \backslash \{x\}$ and $N_y = P \cap R^-_{\pi/4}(y, P) \backslash \{y\}$. It follows from our identities that, $\deg^-_{\pi/4}(x) = |N_x|$ and $\deg^-_{\pi/4}(y) = |N_y|$. Furthermore, by the Inclusion-Exclusion Principle for finite sets, we have $|N_x| + |N_y| = |N_x \cup N_y| + |N_x \cap N_y|$. We shall prove by contradiction that $|N_x \cap N_y| \leq 1$. Suppose that there are two points $u, v \in N_x \cap N_y$. This means that there are points $u_x, v_x, u_y, v_y \in P$ such that the triangles $xu_xu, xv_xv, yu_yu, yv_yv$ are IRTs oriented counterclockwise with right angle at either u or v.

But notice that the line segments u_xu_y and v_xv_y are simply the $(\pi/2)$-counterclockwise rotations of xy about centers u and v respectively. Hence, $u_xu_yv_xv_y$ is a parallelogram with two sides having length xy as shown in Figure 2(a). This is a contradiction since one of



(a) (b)

Figure 2: Proof of Theorem 4.

the diagonals of the parallelogram is longer than any of it sides. Thus, $|N_x \cap N_y| \leq 1$. Furthermore, $x \notin N_y$ and $y \notin N_x$, so $|N_x \cup N_y| \leq n-2$ and thus

$$\deg^-_{\pi/4}(x) + \deg^-_{\pi/4}(y) = |N_x \cup N_y| + |N_x \cap N_y| \leq n - 2 + 1 = n - 1.$$

This also implies that

$$\deg^+_{\pi/4}(x) + \deg^+_{\pi/4}(y) \leq n - 1,$$

since we can follow the exact same argument applied to the reflection of P about the line xy.

We now look at $\deg_{\pi/2}(x)$ and $\deg_{\pi/2}(y)$. First we need the following lemma.

Lemma 2. *For every $p \in P$, at most one of $R_{\pi/2}(x, p)$ or $R_{\pi/2}(y, p)$ belongs to P.*

Proof. Let $p_x = R_{\pi/2}(x, p)$ and $p_y = R_{\pi/2}(y, p)$ (see Figure 2(b)). Note that the distance $p_x p_y$ is exactly the distance xy but scaled by $\sqrt{2}$. This contradicts the fact that xy is the diameter of P. □

Let us define a graph G with vertex set $V(G) = P \backslash \{x, y\}$ and where uv is an edge of G, (i.e., $uv \in E(G)$) if and only if $v = R_{\pi/2}(x, u)$ or $v = R_{\pi/2}(y, u)$.

Lemma 3.

$$0 \leq \deg_{\pi/2}(x) + \deg_{\pi/2}(y) - |E(G)| \leq 1.$$

Proof. The left inequality follows from the fact each edge counts an IRT in either $\deg_{\pi/2}(x)$ or $\deg_{\pi/2}(y)$ and possibly in both. However, if uv is an edge of G so that $v = R_{\pi/2}(x, u)$ and $u = R_{\pi/2}(y, v)$, then $xuyv$ is a square, so this can only happen for at most one edge. □

Now, let $\deg_G(u)$ be the number of edges in $E(G)$ incident to u. We prove the following lemma.

Lemma 4. *For every $u \in V(G)$, $\deg_G(u) \leq 2$.*

Proof. Suppose $uv_1 \in E(G)$, see Figure 3(a). Without loss of generality we can assume that $u = R_{\pi/2}(y, v_1)$. If $v_3 = R_{\pi/2}(y, u) \in P$, then we conclude that $xv_3 > xy$ or $xv_1 > xy$, because $\angle xyv_3 \geq \pi/2$ or $\angle xyv_1 \geq \pi/2$. This contradicts the fact that xy is the diameter of P. Similarly, if v_2 and v_4 are defined as $u = R_{\pi/2}(x, v_4)$ and $v_2 = R_{\pi/2}(x, u)$, then at most one of v_2 or v_4 can be in P. □

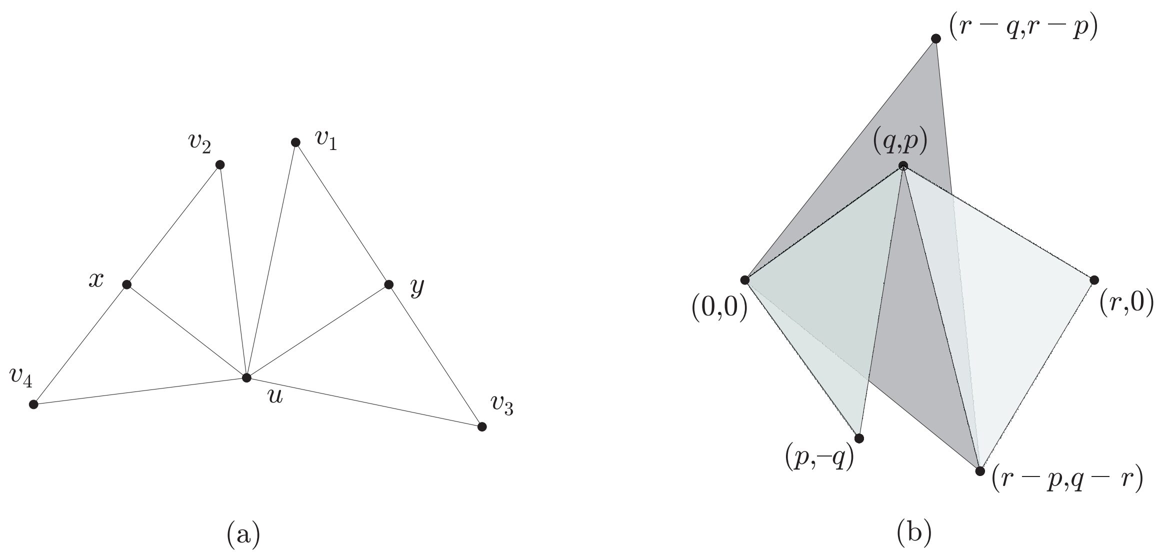


Figure 3: Proof of Lemmas 4 and 5.

We still need one more lemma for our proof.

Lemma 5. *All paths in G have length at most 2.*

Proof. We prove this lemma by contradiction. Suppose we can have a path of length 3 or more. To assist us, let us place our points on a cartesian coordinate system with our diameter xy relabeled as the points $(0,0)$ and $(r,0)$, furthermore, assume $p,q \geq 0$ and that the four vertices of the path of length 3 are $(p,-q)$, (q,p), $(r-p,q-r)$, and $(r-q,r-p)$. Our aim is to show that the distance between $(r-q,r-p)$ and $(r-p,q-r)$ contradicts that r is the diameter of P. Now, if paths of length 3 were possible, then the distance between every pair of points in Figure 3(b) must be less than or equal to r. Since $d((p,-q),(q,p)) \leq r$ then $p^2+q^2 \leq r^2/2$.

Now let us analyze the square of the distance from $(r-q,r-p)$ to $(r-p,q-r)$. Because $2(p^2+q^2) \geq (p+q)^2$, it follows that

$$\begin{aligned} d^2((r-q,r-p),(r-p,q-r)) &= (-q+p)^2 + (2r-p-q)^2 \\ &= 4r^2 - 4r(p+q) + 2(p^2+q^2) \\ &\geq 4r^2 - 4\sqrt{2}r\sqrt{p^2+q^2} + 2(p^2+q^2) = \left(2r - \sqrt{2(p^2+q^2)}\right)^2. \end{aligned}$$

But $\sqrt{2(p^2+q^2)} \leq r$, so $(2r - \sqrt{2(p^2+q^2)}) \geq r$ and thus

$$d^2((r-q,r-p),(r-p,q-r)) \geq r^2.$$

Equality occur if and only if $p = r/2$ and $q = r/2$; otherwise, $d((r-q,r-p),(r-p,q-r))$ is strictly greater than r, contradicting the fact that the diameter of P is r. Therefore if $p \neq r/2$ or $q \neq r/2$ then there is no path of length 3. In the case that $p = r/2$ and $q = r/2$ the points (q,p) and $(r-q,r-p)$ become the same and so do the points $(p,-q)$ and $(r-p,q-r)$. Thus we are left with a path of length 1. ∎

It follows from Lemmas 4 and 5 that all paths of length 2 are disjoint. In other words, G is the union of disjoint paths of length less than or equal to 2. Let a denote the number of paths of length 2 and b denote the number of paths of length 1, then

$$|E(G)| = 2a + b \text{ and } 3a + 2b \leq n - 2.$$

Recall from Lemma 3 that either $\deg_{\pi/2}(x) + \deg_{\pi/2}(y) = |E(G)|$ or $\deg_{\pi/2}(x) + \deg_{\pi/2}(y) = |E(G)| + 1$. If $\deg_{\pi/2}(x) + \deg_{\pi/2}(y) = |E(G)|$, then

$$2\,|E(G)| = 4a + 2b \leq n - 2 + a \leq n - 2 + \frac{n-2}{3},$$

so $\deg_{\pi/2}(x)+\deg_{\pi/2}(y) = |E(G)| \leq \frac{2}{3}\,(n-2)$. Moreover, if $\deg_{\pi/2}(x)+\deg_{\pi/2}(y) = |E(G)|+1$, then $b \geq 1$ and we get a minor improvement,

$$2\,|E(G)| = 4a + 2b \leq n - 2 + a \leq n - 4 + \frac{n-2}{3},$$

so $\deg_{\pi/2}(x) + \deg_{\pi/2}(y) = |E(G)| + 1 \leq (2n-7)\,/3 < \frac{2}{3}\,(n-2)$.

We are now ready to put everything together. Between the two points x and y, we derived the following bounds:

$$\begin{aligned}
&\deg_{\pi/2}(x) + \deg_{\pi/2}(y) \leq \frac{2}{3}(n-2),\\
&\deg^{+}_{\pi/4}(x) + \deg^{+}_{\pi/4}(y) \leq (n-1), \text{ and}\\
&\deg^{-}_{\pi/4}(x) + \deg^{-}_{\pi/4}(y) \leq (n-1).
\end{aligned}$$

Because the degree of a point must take on an integer value, it must be the case that either x or y satisfies $\deg_{\pi/2} + \deg^{+}_{\pi/4} + \deg^{-}_{\pi/4} \leq \lfloor (4n-5)/3 \rfloor$. □

4 Small Cases

In this section we determine the exact values of $S_{\triangle}(n)$ when $3 \leq n \leq 9$.

Theorem 5. *For $3 \leq n \leq 9$, $S_{\triangle}(3) = 1$, $S_{\triangle}(4) = 4$, $S_{\triangle}(5) = 8$, $S_{\triangle}(6) = 11$, $S_{\triangle}(7) = 15$, $S_{\triangle}(8) = 20$, and $S_{\triangle}(9) = 28$.*

Proof. We begin with $n = 3$. Since 3 points uniquely determine a triangle, and there is an IRT with 3 points (Figure 5(a)), this situation becomes trivial and we therefore conclude that $S_{\triangle}(3) = 1$.

Now let $n = 4$. In Figure 5(b) we exhibit a point-set P such that $S_{\triangle}(P) = 4$. This implies that $S_{\triangle}(4) \geq 4$. However, $S_{\triangle}(4)$ is also bounded above by $\binom{4}{3} = 4$. Hence, $S_{\triangle}(4) = 4$.

To continue with the proof for the remaining values of n, we need the following two lemmas.

Lemma 6. *Suppose $|P| = 4$ and $S_{\triangle}(P) \geq 2$. The sets in Figure 5(b)–(e), not counting symmetric repetitions, are the only possibilities for such a set P.*

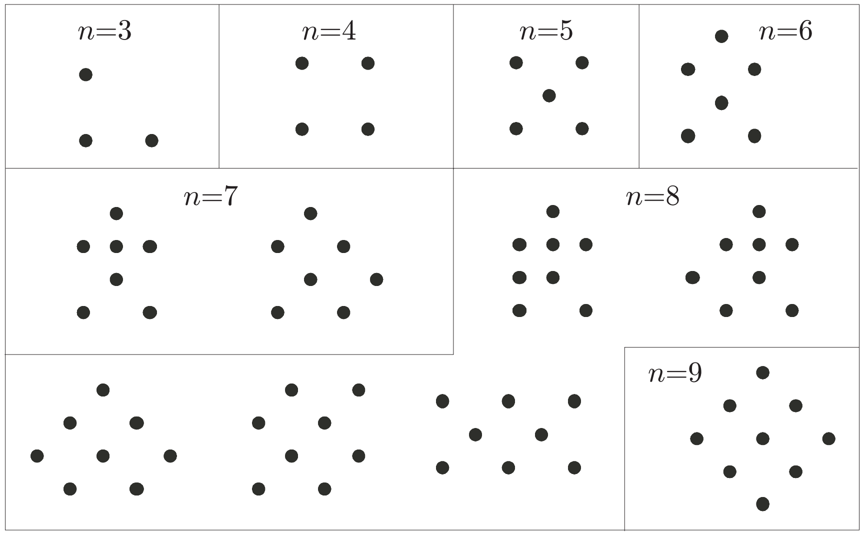


Figure 4: Optimal sets achieving equality for $S_{\triangle}(n)$.

Proof. Having $S_{\triangle}(P) \geq 2$ implies that we must always have more than one IRT in P. Hence, we can begin with a single IRT and examine the possible ways of adding a point and producing more IRTs. We accomplish this task in Figure 5(a). The 10 numbers in the figure indicate the location of a point, and the total number of IRTs after its addition to the set of black dots. All other locations not labeled with a number do not increase the number of IRTs. Therefore, except for symmetries, all the possibilities for P are depicted in Figures 5(b)–(e). □

Lemma 7. *Let P be a finite set with $|P| = n$. Suppose that $S_{\triangle}(A) \leq b$ for all $A \subseteq P$ with $|A| = k$. Then*

$$S_{\triangle}(P) \leq \left\lfloor \frac{n\,(n-1)\,(n-2)\,b}{k\,(k-1)\,(k-2)} \right\rfloor.$$

Proof. Suppose that within P, every k-point configuration contains at most b IRTs. The number of IRTs in P can then be counted by adding all the IRTs in every k-point subset of P. However, in doing so, we end up counting a fixed IRT exactly $\binom{n-3}{k-3}$ times. Because $S_{\triangle}(A) \leq b$ we get,

$$\binom{n-3}{k-3} S_{\triangle}(P) = \sum_{\substack{A \subseteq P \\ |A|=k}} S_{\triangle}(A) \leq \binom{n}{k} b.$$

Notice that $S_{\triangle}(P)$ can only take on integer values so,

$$S_{\triangle}(P) \leq \left\lfloor \frac{\binom{n}{k} b}{\binom{n-3}{k-3}} \right\rfloor = \left\lfloor \frac{n\,(n-1)\,(n-2)\,b}{k\,(k-1)\,(k-2)} \right\rfloor. \qquad \square$$

Now suppose $|P| = 5$. If $S_{\triangle}(A) \leq 1$ for all $A \subseteq P$ with $|A| = 4$, then by Lemma 7, $S_{\triangle}(P) \leq 2$. Otherwise, by Lemma 6, P must contain one of the 4 sets shown in Figures

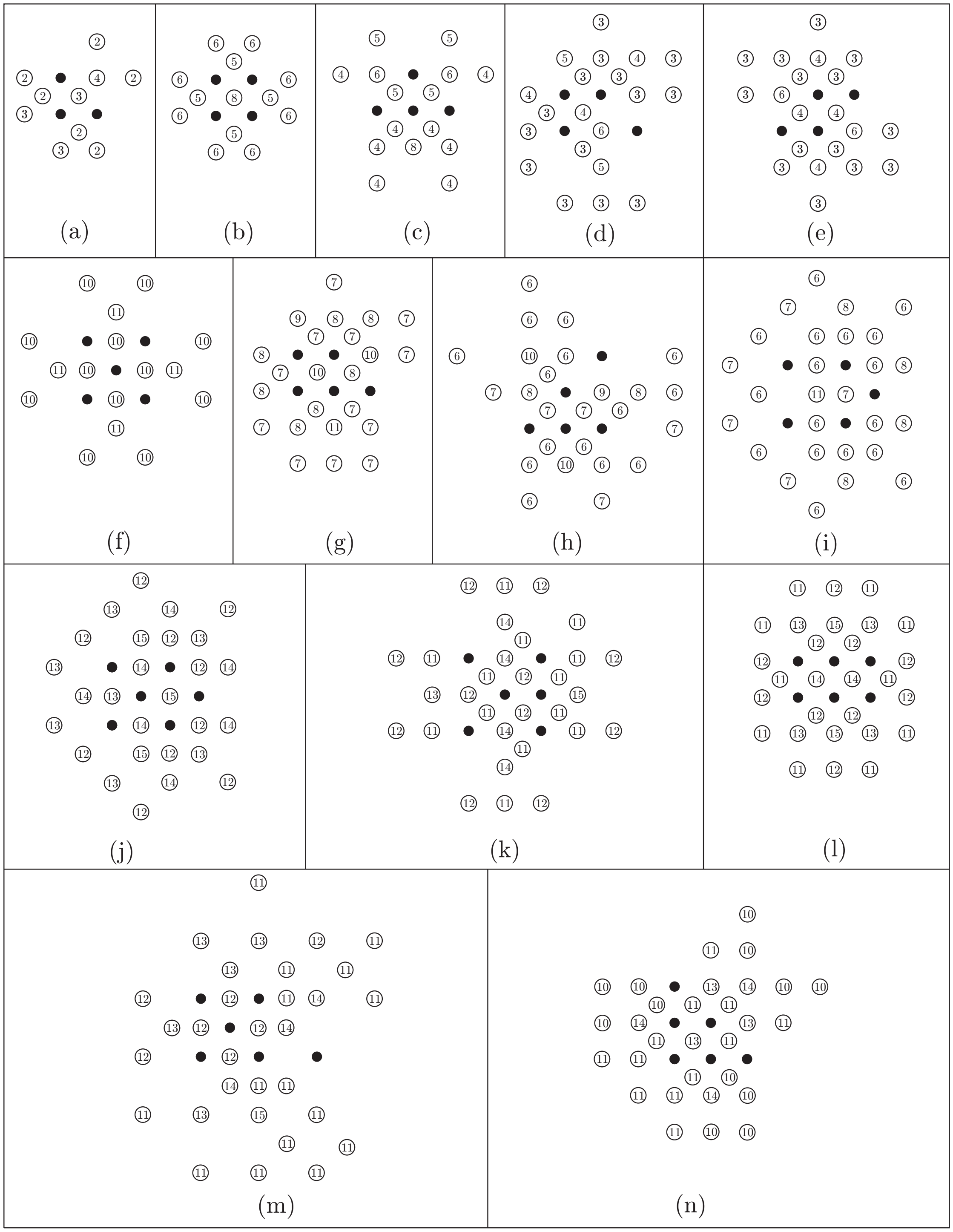


Figure 5: Proof of Theorem 5. Each circle with a number indicates the location of a point and the total number of IRTs resulting from its addition to the base set of black dots.

5(b)–5(e). The result now follows by examining the possibilities for producing more IRTs by placing a fifth point in the 4 distinct sets. In Figures 5(b), 5(c), 5(d), and 5(e) we accomplish this task. In the same way as we did in Lemma 6, every number in a figure indicates the location of a point, and the total number of IRTs after its addition to the set of black dots. It follows that the maximum value achieved by placing a fifth point is 8 and so $S_{\triangle}(5) = 8$. The point-set that uniquely achieves equality is shown in Figure 5(f). Moreover, there is exactly one set P with $S_{\triangle}(P) = 6$ (shown in Figure 5(g)), and two sets P with $S_{\triangle}(P) = 5$ (Figures 5(h) and 5(i)).

Now suppose $|P| = 6$. If $S_{\triangle}(A) \leq 4$ for all $A \subseteq P$ with $|A| = 5$, then by Lemma 7, $S_{\triangle}(P) \leq 8$. Otherwise, P must contain one of the sets in Figures 5(f)–5(i). We now check all possibilities for adding more IRTs by joining a sixth point to our 4 distinct sets. This is shown in Figures 5(f)–5(i). It follows that the maximum value achieved is 11 and so $S_{\triangle}(6) = 11$. The point-set that uniquely achieves equality is shown in Figure 5(j). Also, except for symmetries, there are exactly 3 sets P with $S_{\triangle}(P) = 10$ (Figures 5(k)–5(m)) and only one set P with $S_{\triangle}(P) = 9$ (Figure 5(n)).

Now suppose $|P| = 7$. If $S_{\triangle}(A) \leq 8$ for all $A \subseteq P$ with $|A| = 6$, then by the Lemma 7, $S_{\triangle}(P) \leq 14$. Otherwise, P must contain one of the sets in Figures 5(j)–5(n). We now check all possibilities for adding more IRTs by joining a seventh point to our 5 distinct configurations. We complete this task in Figures 5(j)–5(n). Because the maximum value achieved is 15, we deduce that $S_{\triangle}(7) = 15$. In this case, there are exactly two point-sets that achieve 15 IRTs.

The proof for the values $n = 8$ and $n = 9$ follows along the same lines, but there are many more intermediate sets to be considered. We omit the details. All optimal sets are depicted in Figure 4. □

Inspired by our method used to prove exact values of $S_{\triangle}(n)$, a computer algorithm was devised to construct the best 1-point extension of a given base set. This algorithm, together with appropriate heuristic choices for some initial sets, lead to the construction of point sets with many IRTs giving us our best lower bounds for $S_{\triangle}(n)$ when $10 \leq n \leq 25$. These lower bounds are shown in Table 1 and the point-sets achieving them in Figure 6.

n	10	11	12	13	14	15	16	17
$S_{\triangle}(n) \geq$	35	43	52	64	74	85	97	112

n	18	19	20	21	22	23	24	25
$S_{\triangle}(n) \geq$	124	139	156	176	192	210	229	252

Table 1: Best lower bounds for $S_{\triangle}(n)$.

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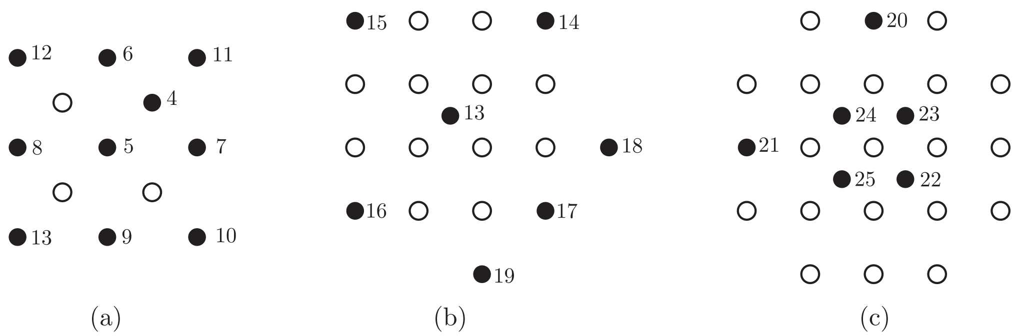

Figure 6: Best constructions A_n for $n \leq 25$. Each set A_n is obtained as the union of the starting set (in white) and the points with label $\leq n$. The value $S_{\triangle}(A_n)$ is given by Table 1.

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