On $(\leq k)$–pseudoedges in generalized configurations and the pseudolinear crossing number of K_n

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July 18, 2006

Abstract

It is known that every generalized configuration with n points has at least $3\binom{k+2}{2}$ $(\leq k)$–pseudoedges, and that this bound is tight for $k \leq n/3 - 1$. Here we show that this bound is no longer tight for (any) $k > n/3 - 1$. As a corollary, we prove that the usual and the pseudolinear (and hence the rectilinear) crossing numbers of the complete graph K_n are different for every $n \geq 10$. It has been noted that all known optimal rectilinear drawings of K_n share a triangular–like property, which we abstract into the concept of 3–decomposability. We give a lower bound for the crossing numbers of all pseudolinear drawings of K_n that satisfy this property. This bound coincides with the best general lower bound known for the rectilinear crossing number of K_n, established recently in a groundbreaking work by Aichholzer, García, Orden, and Ramos. We finally use these results to calculate the pseudolinear (which happen to coincide with the rectilinear) crossing numbers of K_n for $n \leq 12$ and $n = 15$.

1 Introduction

Recently, Ábrego and Fernández–Merchant [1] (see also the closely related work by Lovász, Vesztergombi, Wagner, and Welzl [19]), unveiled and exploited the close connection between the number $e_k(S)$ of k–pseudoedges in a generalized configuration S of n points in the plane, and the crossing number of the (pseudolinear) drawing of K_n defined by S. We recall that if S is a generalized configuration with n points, a k*–pseudoedge* is a pseudosegment (which, we recall, spans 2 points of S) that separates k points from the remaining $n-k-2$ points. A $(\leq k)$*–pseudoedge* is an i–pseudoedge with $i \leq k$, and $E_k(S) := \sum_{0 \leq i \leq k} e_i(S)$ denotes the number of $(\leq k)$–pseudoeges in S.

We emphasize that although in [19] the focus is on geometrical (which may be regarded as particular generalized) configurations, most results in [19] (more precisely, everything except Section 4) are easily translated from k–edges into k–pseudoedges.

A generalized configuration S of n points naturally induces a *pseudolinear drawing* $\mathcal{D}_S$ of K_n, by letting the pseudoedges represent the edges of K_n. The *crossing number* $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ of $\mathcal{D}_S$ is the number of crossings of pseudoedges in $\mathcal{D}_S$, and the *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$ of K_n is the minimum of $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ taken over all generalized configurations S with n points. Since every rectilinear drawing of K_n is pseudolinear, it

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follows that the rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$ of K_n satisfies $\mathrm{cr}(K_n) \leq \widetilde{\mathrm{cr}}(K_n) \leq \overline{\mathrm{cr}}(K_n)$, where $\mathrm{cr}(K_n)$ denotes the usual crossing number of K_n. It is not known whether the second inequality is always tight (we conjecture it is; see Section 7).

In both [1] and [19], the central results are: (i) an expression of $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ in terms of the number $e_k(S)$ of k–pseudosegments of S; and (ii) a lower bound for $E_k(S)$ (and thus indirectly for $e_k(S)$) for any generalized configuration S and any $k < (n-2)/2$. The expressions are given in Lemma 4 in [1] and Lemma 5 in [19]:

$$\widetilde{\mathrm{cr}}(\mathcal{D}_S) = \sum_{k<\frac{n-2}{2}} e_k(S)\left(\frac{n-2}{2} - k\right)^2 - \frac{3}{4}\binom{n}{3}. \tag{1}$$

The general bound for $E_k(S)$ derived in both [1] and [19] is the same in both cases, namely

$$E_k(S) \geq 3\binom{k+2}{2}. \tag{2}$$

Using these results, it is proved in [1] (and follows from the work in [19]) that $\widetilde{\mathrm{cr}}(K_n) \geq (3/8)\binom{n}{4} + O(n^3)$. Lovász et al. improved the coefficient to $3/8 + \epsilon$, where $\epsilon \approx 10^{-5}$ for geometrical configurations. The value of ϵ was subsequently improved by Balogh and Salazar, for generalized configurations, to $\epsilon \approx 5.3 \times 10^{-4}$ [12]. Although these may appear to be marginal improvements at first sight, they are actually quite substantial: the (usual) crossing number $\mathrm{cr}(K_n)$ is known to be at most $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor = (3/8)\binom{n}{4} + O(n^3)$. Thus these improvements show that (and estimate for how much) that the usual and the pseudolinear (and hence rectilinear) crossing numbers of K_n differ in the asymptotically relevant term. Thus the best general lower bound reported in the literature so far is $\widetilde{\mathrm{cr}}(K_n) \geq 0.37553\binom{n}{4} + O(n^3)$ [12]. Regarding upper bounds, the best general known upper bound is $\widetilde{\mathrm{cr}}(K_n) \leq \overline{\mathrm{cr}}(K_n) \leq 0.38056\binom{n}{4} + O(n^3)$ [2].

Our aim in this work is to present some new results on the aforementioned problems.

First we focus on the currently best known general lower bound $E_k(S) \geq 3\binom{k+2}{2}$. This bound is known to be tight for every $k \leq n/3$ (attained, in particular, in rectilinear drawings), and from [12] it follows that it is not tight for some $k > n/3$ (especially for k close to $n/2$). We have completely settled the question of tightness for this inequality, as follows.

Theorem 1 *Let S be a generalized configuration of n points in the plane. For every $k > n/3$,*

$$E_k(S) > 3\binom{k+2}{2}.$$

It has been reported in the literature, alternately as a conjecture and as a fact, that $\overline{\mathrm{cr}}(K_n)$ and $\mathrm{cr}(K_n)$ are different for every $n \geq 10$ (see for instance [11]). After a quite exhaustive search, no proof seems to have been published. As a corollary of Theorem 1, we are able to show that this is indeed the case, not only for the rectilinear crossing number, but for the pseudolinear crossing number as well.

Corollary 2 *The pseudolinear (and, consequently, the rectilinear) crossing number of K_n is strictly greater than the usual crossing number of K_n for every $n \geq 10$.*

We then move on to an analysis motivated by all known optimal pseudolinear (as it happens, rectilinear) drawings of K_n. Even the most superficial glance at these drawings reveals that the n points (vertices) are grouped into clusters of size $n/3$ (or as close as possible to $n/3$ if 3 does not divide n). This property is naturally captured and formalized into the realm of generalized configurations (and hence pseudolinear drawings) in terms of circular sequences (see next section) in what we call *3–decomposability*.

For the reader familiar with circular sequences and their relationship with k–pseudoedges, and the ongoing notation and terminology associated to them, let us now formally state these results. The reader not familiar with these may skip this discussion and come back after picking up the necessary concepts in Section 2. Let us say that a (doubly infinite) circular sequence $\mathbf{\Pi}$ is *3–decomposable* if there is an n–halfperiod Π such that if $(a_m, a_{m-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots c_m)$ is the first permutation of Π then all transpositions between an element of $A = \{a_m, a_{m-1}, \ldots, a_1\}$ and an element of $B = \{b_1, b_2, \ldots, b_m\}$ occur prior to all transpositions between $C = \{c_1, c_2, \ldots, c_m\}$ and $A \cup B$. In this case we say that Π is *decomposed into classes* A, B, and C. Our main result on 3–decomposability is the following lower bound on the number of $(\leq k)$–pseudosegments for 3–decomposable sequences.

Theorem 3 *If* $\mathbf{\Pi}$ *is a circular sequence associated to a generalized configuration* S *of* $n = 3r$ *points, and* $\mathbf{\Pi}$ *is* 3*–decomposable, then for all* $k < (n-2)/2$*,*

$$E_k(S) \geq 3\binom{k+2}{2} + 3\binom{k-r+2}{2} \qquad \textit{(by convention } \tbinom{p}{q} = 0 \textit{ if } p < q\textit{).}$$

Using this bound and Eq. (1), we immediately obtain the following bound for the number of crossings of any 3-decomposable set.

Theorem 4 *If* S *is a* 3*-decomposable set of* $n = 3m$ *points in the plane in general position then*

$$\begin{aligned}\widetilde{\mathrm{cr}}(\mathcal{D}_S) \quad &\geq \quad \begin{cases} \left(41n^4 - 324n^3 + 846n^2 - 972n + 729\right)/2592 \textit{ if } n \textit{ is odd} \\[2ex] \left(41n^4 - 324n^3 + 792n^2 - 648n\right)/2592 \textit{ if } n \textit{ is even.}\end{cases} \\ &= \quad \frac{41}{2592}n^4 + O\left(n^3\right) = \frac{41}{108}\binom{n}{4} + O\left(n^3\right) = .37\overline{962}\binom{n}{4} + O\left(n^3\right).\end{aligned}$$

We note that this bound coincides with the best general lower bound known for the rectilinear crossing number of K_n, established quite recently in a groundbreaking work by Aichholzer, García, Orden, and Ramos [6].

As we observed above, a motivation for introducing the concept of 3–decomposability is that all known optimal rectilinear drawings of K_n (their associated circular sequences, that is) satisfy this condition. Note that the bound in Theorem 4 is very close to the best general upper bound reported in [2] (see Corollary 15 in Section 4).

Our final application has to do with the exact calculation of the pseudolinear crossing number of K_n. The exact rectilinear crossing number of K_n has been determined for $n \leq 17$ in computer–assisted proofs as part of a wider project of classification of abstract order data types; see [4, 5, 7]. Our final observation in Section 6 is how to apply our results and techniques to provide a computer–free verification of the pseudolinear crossing numbers of K_n for $n \leq 12$ and $n = 15$, which, we observe, coincide with their respective rectilinear crossing numbers. We have used our techniques to compute the pseudolinear crossing number of K_{13}, but currently the details are too lengthy and cumbersome to be worth including here. Regarding K_{14}, there is an interesting, serious obstacle: in order to calculate $\widetilde{\mathrm{cr}}(K_{14})$ (at least with our current techniques) we would need the exact value of $\widetilde{h}(14)$, the maximum number of halving pseudolines in a generalized configuration with 14 points (see Section 6 for further details). This is an open, difficult, and important problem by itself.

In Section 7 we present a couple of open problems.

2 Background: circular sequences and $(\leq k)$–pseudoedges

The main ingredients of the proofs ([1], [19]) that $E_k(S) \geq 3\binom{k+2}{2}$ are, on the one hand, the close relationship between $(\leq k)$–pseudoedges and circular sequences, and, on the other hand, a careful analysis on (the number of) $(\leq k)$–critical tranpositions in circular sequences.

We recall that a *circular sequence* $\mathbf{\Pi}$ *on* n *elements* is a doubly infinite sequence $(\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ of permutations of the set $\{1, 2, \ldots, n\}$, where any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions, and such that for every ℓ, $\pi_{\ell+\binom{n}{2}}$ is the reverse permutation of π_ℓ (thus $\mathbf{\Pi}$ has period $2\binom{n}{2}$). A transposition that occurs between elements in positions i and $i+1$, or between elements in positions $n-i$ and $n-i+1$ is *i–critical*. A transposition is *$(\leq k)$–critical* if it is critical for some $i \leq k$. We denote the number of $(\leq k)$–critical transpositions in a finite subsequence Π of $\mathbf{\Pi}$ by $\chi_{\leq k}(\Pi)$.

Circular sequences, introduced by Goodman and Pollack in an influential paper [18], are a fruitful tool to encode generalized configurations of points by means of the following construction (for simplicity, suppose that S is a set of n points in general position, and consider the geometrical configuration defined by S and the straight lines defined by the pairs of points in S). Let L be a (directed) line that is not orthogonal to any of the lines defined by pairs of points in S. We label the points in S as $p_1, p_2, \ldots, p_n$, according to the order in which their orthogonal projections appear along L. As we rotate L (say counterclockwise), the ordering of the projections changes precisely at the positions where L passes through a position orthogonal to the line defined by some pair of points r, s in S. At the time the projection change occurs, r and s are adjacent in the ordering. and the ordering changes by transposing r and s. By keeping track of all permutations of the projections as L is rotated (clockwise and counterclockwise), we obtain the circular sequence on n elements *associated to* S.

The crucial observation is the following, implicit in [1] in its full generality (generalized configurations), and explicit in [19] for ordinary configurations.

Proposition 5 *Suppose that* $\mathbf{\Pi}$ *is the circular sequence associated to a generalized configuration* S *of* n *points in general position. Let* Π *be any halfperiod of* $\mathbf{\Pi}$*. Then the* $(\leq k)$*–pseudoedges of* S *are in one–to–one correspondence with* $(\leq k+1)$*–critical transpositions of* Π*. That is,* $E_k(S) = \chi_{\leq k+1}(\Pi)$*.*

This transforms the geometrical problem of counting $(\leq k)$–pseudoedges into the combinatorial problem of counting $(\leq k)$–critical transpositions of halfperiods of circular sequences. For brevity, we shall call a halfperiod of a circular sequence on n elements an *n–halfperiod*.

In order to explain the approach and give the main results in [19] on $(\leq k)$–critical transpositions in n–halfperiods, in what follows we shall closely follow the lively notation and terminology introduced in that paper.

Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be an n–halfperiod. For each $k < n/2$, define $m = m(k, n) := n - 2k$. In order to keep track of $(\leq k)$–critical transpositions in Π, it is convenient to label the points so that the starting permutation is $\pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k)$.

An element x *exits* (respectively, *enters*) *through the i–th A–gate* if it moves from position $k-i+1$ to position $k-i+2$ (respectively, from position $k-i+2$ to position $k-i+1$) during a transposition with another element. Similarly, x *exits* (respectively, *enters*) *through the i–th C–gate* if it moves from position $m+k+i$ to position $m+k+i-1$ (respectively, from $m+k+i-1$ to $m+k+i$) during a transposition.

An $a \in \{a_1, \ldots, a_k\}$ (respectively, $c \in \{c_1, \ldots, c_k\}$) is *confined* until the first time it exits through the first A–gate (respectively, C–gate); then it becomes *free*. A transposition is *confined* if both elements involved are confined. An important simplifying observation in [19] is the following.

Proposition 6 *Let Π_0 be an n–halfperiod, and let $k < n/2$. Then there is an n–halfperiod Π, with the same number of $(\leq k)$–critical transpositions as Π_0, and with no confined transpositions.*

In view of this statement, for the rest of the section we assume that the n–halfperiod Π under consideration has no confined transpositions.

The *liberation sequence* $\sigma(\Pi)$ (or simply σ if no confusion arises) of Π contains all the a's and all the c's, in the order in which they become free in Π. Since Π has no confined transpositions, the a's appear in increasing order, as do the c's. We let $T(a_i)$ (respectively $T(c_i)$) denote the set of of all those c's (respectively a's) that appear after a_i (respectively c_i) in σ.

A transposition that swaps elements in positions i and $i+1$ occurs in the *A–Zone* (respectively, *C–Zone*) if $i \leq k$ (respectively, $i \geq k+m$). Such transpositions are of obvious relevance: a transposition is $(\leq k)$–critical iff it occurs either in the A–Zone or in the C–Zone.

For $1 \leq i \leq j \leq k$, the i–th A–gate is a *compulsory exit gate* for a_j, and the i–th C–gate is a *compulsory entry gate* for a_j: that is, a_j has to exit through the i–th A–gate at least once, and to enter the i–th C–gate at least once. Analogous definitions and observations hold for c_j: the i–th A–gate is a *compulsory entry gate* for c_j, and the i–th C–gate is a *compulsory exit gate* for c_j. A transposition in which an element enters (respectively, exits) one of its compulsory entry (respectively, exit) gates for the first time is a *discovery* transposition *for* the element. A transposition is a *discovery* transposition if it is a discovery transposition for at least one of the elements involved. If it is a discovery transposition for both elements, then it is a *double–discovery* transposition (for the reader familiar with [19], what we call double–discovery transpositions are the transpositions represented by a directed edge in the *savings digraph* of [19]).

Discovery and double–discovery transpositions play a central role in [19]. The key results are the following, which hold for any n–halfperiod with no confined transpositions (the first statement is a straightforward counting, whereas the second does definitely require a proof). Both results are in [19].

Proposition 7 *There are (exactly) $2\binom{k+1}{2}$ transpositions that are discovery transpositions for some a, and (exactly) $2\binom{k+1}{2}$ transpositions that are discovery transpositions for some c.*

Proposition 8 *There are at most $\binom{k+1}{2}$ double–discovery transpositions.*

Since each discovery transposition is $(\leq k)$–critical, these statements immediately imply the following.

Proposition 9 *There are at least $3\binom{k+1}{2}$ $(\leq k)$–critical transpositions.*

We note that $E_k(S) \geq 3\binom{k+1}{2}$ follows from Propositions 5 and 9.

To push this one step further for the case $k > n/3$, we need a deeper look at the proof of Theorem 10 in [19] (where Propositions 7 and 8 are established).

Definition An n–halfperiod Π with no confined transpositions is *perfect* if the following hold:

(a) Each transposition in Π that occurs in the A–Zone or in the C–Zone is a discovery transposition.

(b) Each a_i is involved in (exactly) $\min\{i, |T(a_i)|\}$ double–discovery transpositions in the C–Zone.

(c) Each c_i is involved in (exactly) $\min\{i, |T(c_i)|\}$ double–discovery transpositions in the A–Zone.

The following result is implicit in the proof of Theorem 10 in [19].

Proposition 10 *If Π is perfect, then it has exactly $3\binom{k+1}{2}$ $(\leq k)$–critical tranpositions. Conversely, if Π has no confined transpositions, and has exactly $3\binom{k+1}{2}$ $(\leq k)$–critical transpositions, then it is perfect.*

Our approach to prove Theorem 1 is to take a more careful look at n–halfperiods. The main tool is Proposition 13 below: if Π is perfect, then $m \geq k$. Theorem 1 follows immediately by combining this result with Proposition 10.

3 Proof of Theorem 1

Our goal in this section is to show that if Π is perfect, then $m \geq k$ (Proposition 13). This is a key result, from which Theorem 1 follows easily (see end of this section). The main ingredient in the proof of Proposition 13 is a complete characterization of the possible liberation sequences that a perfect Π can have.

Proposition 11 *Suppose that Π is perfect. Then, in the liberation sequence σ of Π, either all the a's occur consecutively or all the c's occur consecutively.*

Proof. The last entry in σ is either a_k or c_k, and by symmetry we may assume without any loss of generality that it is a_k. Our strategy is to suppose that $a_{t-1}c_\ell c_{\ell+1} \cdots c_k a_t \cdots a_k$ is a suffix of σ, where $\ell > 1$ and $2 \leq t \leq k$, and derive a contradiction.

We claim that a_{t-1} swaps with c_k in the C–Zone. We start by noting that since Π is perfect, and $|T(a_{t-1})| = k - \ell + 1 \geq 1$, it follows that a_{t-1} is involved in a double–discovery transposition in the C–zone with at least one c. If this transposition involves (a_{t-1} and) c_k, then our claim obviously holds. Thus suppose that it involves (a_{t-1} and) c_i for some $i < k$. Then, right after a_{t-1} and c_i swap, c_k is to the right of a_{t-1}, since no confined transpositions occur in Π. Note that all transpositions that swap a_{t-1} to the left involve an a_j with $j > t-1$. On the other hand, since a_t (moreover, every a_j with $j \geq t$) gets freed after c_k, it follows that before any transposition can move a_{t-1} left, c_k must be freed (and before that it must transpose with a_{t-1}). This shows that the transposition μ that swaps a_{t-1} with c_k occurs in the C–Zone.

Thus, right after μ occurs, a_{t-1} is at position r, where $r \geq k+m+1$. We claim that $\max\{r, k+m+t-1\} < 2k+m$. Since $t-1 < k$, then $k+m+t-1 < 2k+m$, and so it suffices to show that if $r > k+m+t-1$, then $r < 2k+m$. So suppose that $r > k+m+t-1$. Note that the final position in Π (that is, the position in $\pi_{\binom{n}{2}}$) of a_{t-1} is $k+m+t-1$, and so by the time μ occurs there has been a transposition τ that moves a_{t-1} to the right of its final position (we remark that possibly $\tau = \mu$). Since τ occurs in the C–Zone and clearly is not a discovery step for a_{t-1}, and Π is perfect, it follows that τ is a discovery step for a c_i. Moreover, $|T(a_{t-1})| = k-\ell+1$ is greater than $t-1$, as otherwise (by the perfectness of Π) the transposition between a_{t-1} and c_i would have to be a double–discovery step. Thus $|T(a_{t-1})| > t-1$, and again invoking the perfectness of Π we get that a_{t-1} is involved with (exactly) $t-1$ double–discovery steps in the C–Zone, each with an element in $\{c_\ell, \ldots, c_k\}$. Therefore the number of possible transpositions that move a_{t-1} to the right of its final position $k+m+t-1$ is at most $k-\ell+1-(t-1)$. Thus the rightmost position of a_{t-1} throughout Π (and consequently r) is at most $k+m+t-1+k-\ell+1-(t-1) = 2k+m+1-\ell < 2k+m$.

Let R be the set of the points that occupy the positions $r+1, r+2, \ldots, 2k+m$ immediately after μ occurs. Since at this time every a_j with $j > t-1$ is confined, it follows that each point in R is either a b, a free c (this follows easily since there are no confined transpositions, and a_{t-1} reached position r by transposing with c_k), or an a_j with $j < t-1$. In particular, each element in R still has to transpose with a_{t-1}.

We claim that a_{t-1} must move back to the B–Zone (after μ occurs). Seeking a contradiction, suppose that this is not the case. We claim that then there is a transposition ρ of a_{t-1} with an element in R that is not a discovery tranposition. For suppose that a_{t-1} does not go back to the B–Zone. The key observation is that then at most $k+m+t-1-r$ transpositions of a_{t-1} with elements of R can be discovery transpositions. In order to prove this assertion, first we note that no transposition of a_{t-1} with an element in R can be discovery transposition for the element in R (recall that each element in R is either a b, a free c, or an a_j with $j < t-1$), so if such a transposition is a discovery one, it is so for a_{t-1} (recall we assume that a_{t-1} does not go back to the B–Zone). But once a_{t-1} has reached r, it has at most $k+m+t-1-r$ discovery transpositions to do (since the rightmost compulsory entry gate for a_{t-1} is the $(t-1)$–st C–gate). Now since R has $2k+m-r$ elements, and $2k+m-r > k+m+t-1-r$, it follows that there is at least one transposition ρ of a_{t-1} with an element of R that is not a discovery transposition, as claimed. But the perfectness of Π implies that such a transposition must occur in the B–Zone, contradicting (precisely) our assumption that a_{t-1} did not move back to the B–Zone.

Thus, after μ occurs, a_{t-1} eventually re–enters the B–Zone, and since its final position is $k+m+t-1$, afterwards it has to re–enter the C–zone via a transposition λ that moves a_{t-1} to the right and an element $x \in R$ to the left. Since λ occurs in the C–Zone, and Π is perfect, then λ must be a discovery transposition. We complete the proof by arriving to a contradiction: λ *cannot* be a discovery transposition. Indeed, λ cannot be discovery for a_{t-1} (since it had already been in the C–Zone), so it must be a discovery step for x. On the other hand, since each $x \in R$ is either a b, a free c, or an a_j with $j < t-1$, λ it follows that λ cannot be a discovery transposition for x either. ∎

Our next statement claims that we can a bit further: there is a perfect n–halfperiod Π' whose liberation sequence has all a's followed by all c's or vice versa.

Proposition 12 *Suppose that* Π *is a perfect n–halfperiod of a (doubly–infinite) circular sequence* $\mathbf{\Pi}$. *Then* $\mathbf{\Pi}$ *contains a perfect n–halfperiod* Π', *with initial permutation* $a'_k a'_{k-1} \ldots a'_1\ b'_1 \ldots b'_m\ c'_1 c'_2 \ldots c'_k$, *and whose liberation sequence is either* $a'_1 a'_2 \ldots a'_k c'_1 c'_2 \ldots c'_k$ *or* $c'_1 c'_2 \ldots c'_k a'_1 a'_2 \ldots a'_k$.

Proof. Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be any perfect n–halfperiod, with initial permutation $\pi_0 = (a_k a_{k-1} \ldots a_1$ $b_1 \ldots b_m c_1 c_2 \ldots c_k)$, and let σ be the liberation sequence asociated to Π. Thus the last entry of σ is either a_k or c_k, and a straightforward symmetry argument shows that we may assume whithout loss of generality that last entry in σ is a_k. If σ is $c_1 c_2 \ldots c_k a_1 a_2 \ldots a_k$, then we are done. Thus we may assume that there is a t, $2 \le t \le k$, such that $a_{t-1}, c_1, c_2, \ldots, c_k, a_t, a_{t+1}, \ldots, a_k$ is a suffix of σ.

In order to define the n–halfperiod Π' claimed by the proposition, we establish some facts regarding Π.

(A) *Let π_{i+1} be the permutation where c_1 becomes free. Then π_i is of the form $(a_k, a_{k-1}, \ldots, a_t,\ d_1,\ d_2, \ldots, d_p c_1, c_2, \ldots c_k)$ where $p = t-1+m$ and each d_j is either a b or a free a.*

Proof of (A). The perfectness of Π readily implies that every transposition in the A–Zone that involves an element in $L := \{a_t, a_{t+1}, \ldots, a_k\}$ is a double–discovery transposition. In paticular, the first element that moves an element in L must involve a c. Therefore, as long as no c becomes free, all the elements in L must stay in their original position. Finally, we observe that when c_1 becomes free, $a_1, a_2, \ldots, a_{t-1}$ are already free, so each d_j is either a b or a free a, as claimed.

(B) *No element in $\{a_k a_{k-1} \ldots a_t d_1, \ldots, d_{t-1}\}$ (these are the elements that are in the A–zone, in the given order, in π_i) leaves the A–Zone before c_k becomes free.*

Proof of (B). Seeking a contradiction, let e be the first element in $\{a_k a_{k-1} \ldots a_t d_1, \ldots, d_{t-1}\}$ that moves out of the A–Zone before c_k becomes free. The perfectness of Π readily implies that the element that takes

e out of the A–Zone is some c_j (where by assumption $j \neq k$). Now right after c_j swaps with e, c_j and c_k are in the A– and C–Zones, respectively. In particular, at this point c_j and c_k have not swapped. Now as we observed above, every transposition in the A–Zone involving an element in L is double–discovery, and so it follows that c_j never gets beyond (to the left of) position $k-j+1$. No matter where the $(c_j, c_k) \mapsto (c_k, c_j)$ transposition occurs, this implies that c_j must at some point be in a position r, with $k-j+1 \leq r \leq k$, and then move (right) to position $r+1$. Now in order to reach its final position, c_j must eventually move back to position r, via some transposition $\varepsilon = (x, c_j) \mapsto (c_j, x)$. Since Π is perfect, and ε occurs in the A–Zone, ε is a discovery transposition. But it clearly cannot be discovery for c_j, since c_j is re–visiting position r. Now $x \in \{a_k, a_{k-1}, \ldots, a_t, d_1, \ldots, d_{t-1}\}$, since these were the elements to the left of c_j when it first entered the A–Zone. Clearly x cannot be a d, since each d is either a b or a free a, and ε must be discovery for x. Thus x must be in $L = \{a_k, a_{k-1}, \ldots, a_t\}$. But this is also impossible, since (see Proof of (A)) every transposition that involves an element in L must be a double–discovery transposition.

(C) *Suppose that two elements that are in the A–Zone (respectively, C–Zone) in π_i transpose with each other in the A–Zone (respectively, C–Zone) after π_i. Then at least one of these elements leaves the A–Zone (respectively, C–Zone) after π_i and before this transposition occurs.*

Proof of (C). First we note that the elements that are in the C–Zone in π_i are $c_1, c_2, \ldots, c_k$, in this order, and that if two of them transpose before at least one of them leaves the C–Zone, this transposition would be confined, contradicting the assumption that Π is perfect. That takes care of the C–Zone part of (C).

Now we recall that the elements that are in the A–Zone in π_i are $a_k, a_{k-1}, \ldots, a_t, d_1, d_2, \ldots, d_{t-1}$, in this order. Suppose that two such elements transpose in the A–Zone after π_i, and that between π_i and this transposition (call it λ) none of them leaves the A–Zone. It follows from the perfectness of Π that, for each a_j, every move of a_j until it leaves the A–Zone must involve some c_ℓ. Thus none of the elements involved in λ can be an a_j, that is, both must be d_j's. But such a transposition would clearly not be discovery (recall that each d is a free a or a b), contradicting the perfectness of Π. This completes the proof of (C).

(D) *After π_i, the elements in the A–Zone leave it in the order $d_{t-1}, d_{t-2}, \ldots, d_1, a_t, \ldots, a_{k-1}, a_k$, and the elements in the C–Zone leave it in the order $c_1, c_2, \ldots, c_k$.*

Proof of (D). This is an immediate corollary of (C).

Now define $\Pi' := (\pi_i, \pi_{i+1}, \ldots, \pi_{\binom{n}{2}} = \pi_0^{-1}, \pi_1^{-1}, \ldots, \pi_{i-1}^{-1}, \pi_i^{-1})$. It is straightforward to check that Π' is an n–halfperiod. Define the relabeling $a_i \mapsto a'_i$ for $i = t, t+1, \ldots, k$; $d_s \mapsto a'_{t-s}$ for $s = 1, \ldots, t-1$; $d_s \mapsto b'_{s-t+1}$ for $s = t, t+1, \ldots, p$; and $c_i \mapsto c'_i$ for $i = 1, \ldots, k$, so that the initial permutation of Π' (namely $\pi_i = (a_k a_{k-1} \ldots a_t d_1 d_2 \ \ldots, d_p c_1 c_2 \ldots c_k)$) is $(a'_k a'_{k-1} \ldots a'_1 b'_1 b'_2 \ldots b'_m c'_1 c'_2 \ldots c'_k)$.

To complete the proof, we check that (i) the liberation sequence of Π' is $c'_1 c'_2 \ldots c'_k a'_1 a'_2 \ldots a'_k$; and that (ii) Π' is perfect. We note that (i) follows immediately from (B) and (D). Now in view of Proposition 10, in order to prove that Π' is perfect it suffices to show that it has no confined transpositions, and that it has exactly $3\binom{k+1}{2}$ $(\leq k)$–critical transpositions. From (C) it follows that Π' has no confined transpositions. On the other hand, an application of Proposition 10 to Π (which is perfect) yields that Π has $3\binom{k+1}{2}$ $(\leq k)$–critical transpositions. The construction of Π' clearly reveals that Π and Π' have the same number of $(\leq k)$–critical transpositions, and so Π' has $3\binom{k+1}{2}$ $(\leq k)$–critical transpositions, as required. ∎

Proposition 13 *If Π is perfect, then $m \geq k$.*

Proof. In view of Propositions 11 and 12, and again invoking straightforward symmetry arguments, we may assume that $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$, with initial permutation $\pi_0 = (a_k a_{k-1} \ldots a_1 b_1 \ldots, b_m c_1 c_2 \ldots c_k)$, and that the liberation sequence σ of Π is $a_1 a_2 \ldots a_k c_1 c_2 \ldots c_k$. To prove the proposition, we suppose that Π is perfect and $m < k$, and derive a contradiction.

We first observe that every discovery transposition of an a in the C–Zone must be a double–discovery transposition. Indeed, $|T(a_i)| = k \geq i$ for every i, and so the perfectness of Π implies that all i discovery transpositions for a_i in the C–Zone have to be double–discovery transpositions.

Let c_i be the first c that enters through the first A–gate, via a transposition τ. Let π_s, π_{s+1} be the permutations involved in this step, so that τ transforms π_s into π_{s+1}.

We assume first $c_i = c_k$, and derive a contradiction. In π_s, c_k occupies the $(k+1)$–st position, and all other c's are to the right of c_k. Since $m < k$, this implies that some c_j is in the C–Zone in π_s. Now since c_k is (obviously) already free when τ occurs, so is c_j. Thus c_j became free and later went back to the C–Zone. This is readily seen to be impossible: the transposition that later brings c_j back to the B–Zone from the C–Zone (in its way to its final position) is not a discovery transposition for c_j, and so by the remark above (every discovery transposition for an a in the C–Zone must be double–discovery) it cannot be a discovery transposition for any a either. Thus, such a transposition is not a discovery transposition, contradicting the perfectness of Π.

We complete the proof by deriving a contradiction from the other possibility, namely $c_i \neq c_k$. Let L be the set of all elements to the left of c_i in π_{s+1}. Since c_i occupies position k in π_{s+1}, it follows that $|L| = k-1$. Every $x \in L$ is either a b or a free a (all a's become free before each c becomes free), and so the perfectness of Π implies that every transposition $(x, c_i) \mapsto (c_i, x)$ in the C–Zone has to be a discovery step for c_i, since it cannot be a discovery transposition for x. Moreover, no transposition (after τ occurs) can move c_i to the right. Indeed, suppose that the first such transposition (say ω) moves c_i from position q to position $q+1$. Then $k - i + 1 \leq q$, since c_i cannot move left beyond position $k - i + 1$ (by the remark above, each transposition that moves c_i to its left has to be a discovery transposition for c_i). Thus, after ω occurs, in order to bring c_i to its final position it eventually needs to be moved left from position $q+1$ to position q. Such a transposition $(x, c_i) \mapsto (c_i, x)$ is not discovery for c_i, contradicting our previous remark. Thus, no transposition after τ occurs can move c_i to the right, as claimed. On the other hand, c_i *does* have to move right (at least once) after τ occurs, since $i \neq k$ implies that right after τ occurs c_i still has to transpose with each c in the (nonempty, since $i \neq k$) set $\{c_{i+1}, \ldots, c_k\}$. This contradiction completes the proof. ∎

Proof of Theorem 1.

Let S be a generalized configuration of n points, and let $k > n/3$. Let $\mathbf{\Pi}$ be its associated circular sequence, and let Π be a halfperiod of $\mathbf{\Pi}$. Then, by Proposition 5, $\chi_{\leq k+1}(\Pi) = E_k(S)$.

Since $k > n/3$, then $m = n - 2k < k$. Thus it follows from Proposition 13 that Π cannot be perfect. Hence, by Proposition 10, $\chi_{\leq k+1}(\Pi) = E_k(S)$ is strictly greater than $3\binom{k+2}{2}$. ∎

4 Pseudolinear crossing numbers of generalized configurations with 3–decomposable circular sequences: proof of Theorem 3

For $i = 1, 2, \ldots, n-1$, let t_i be the number of transpositions occurring between elements in positions i and $i+1$ of an finite subsequence Π of $\mathbf{\Pi}$. Let $v(\Pi) = (t_1, t_2, \ldots, t_{n-1})$. Note that if Π is an n–halfperiod then the sum of the i^{th} and $(n-i)^{th}$ entries of $v(\Pi)$ is the number of i–critical transpositions of Π and therefore $\chi_{\leq k}(\Pi)$ is the sum of the first and last k entries of $v(\Pi)$. Because of this symmetry, two such vectors $(v_1, v_2, \ldots, v_{n-1})$ and $(u_1, u_2, \ldots, u_{n-1})$ are *equivalent* if $v_i + v_{n-i} = u_i + u_{n-i}$ for all $1 \leq i < n/2$. Motivated by our intention to use these vectors to bound $\chi_{\leq k}(\mathbf{\Pi})$ (which we define as $\chi_{\leq k}(\Pi)$ for any n–halfperiod Π), we define the following order: $(v_1, v_2, \ldots, v_{n-1}) \preceq (u_1, u_2, \ldots u_{n-1})$ if for all $1 \leq k < n/2$, $\sum_{i=1}^{k}(v_i + v_{n-i}) \leq \sum_{i=1}^{k}(u_i + u_{n-i})$.

In this way, if $\mathbf{\Pi_1}$ and $\mathbf{\Pi}_2$ are circular sequences with n–halfperiods Π_1 and Π_2, respectively, such that $v(\Pi_1) \preceq v(\Pi_2)$ then $\chi_{\leq k}(\mathbf{\Pi_1}) \leq \chi_{\leq k}(\mathbf{\Pi_2})$ for all $1 \leq k < n/2$.

We extend all previous notation to the following kind of sequence of permutations. Given an $(n_a + n_b)$–tuple $(a, a, \ldots, a, b, b, \ldots, b)$ of n_a $a's$ and n_b $b's$, we consider the *bichromatic sequences of permutations* $\Pi' = (\pi'_0, \pi'_1, \ldots, \pi'_{n_a n_b})$ of $(a, a, \ldots, a, b, b, \ldots, b)$ such that $\pi'_0 = (a, a, \ldots, a, b, b, \ldots, b)$ and for $1 \leq i \leq n_a n_b$ the permutation π'_i is obtained from the previous π'_{i-1} by a transposition of two consecutive elements, the first an a and the second a b, This implies that $\pi'_{n_a n_b} = (b, b, \ldots, b, a, a, \ldots, a)$. The following lemma plays a crucial role in the proof of Theorem 3.

Lemma 14 *If* $\Pi' = (\pi'_0, \pi'_1, \ldots, \pi'_{n_a n_b})$ *is a sequence of permutations of* $(\underbrace{a, a, \ldots, a}_{n_a}, \underbrace{b, b \ldots, b}_{n_b})$ *then*

$$v(\Pi') = (1, 2, \ldots, \min(n_a, n_b) - 1, \min(n_a, n_b), \ldots, \min(n_a, n_b), \min(n_a, n_b) - 1, \ldots, 2, 1).$$

Proof of Lemma. Since the initial permutation is $\pi'_0 = (a, a, \ldots, a, b, b, \ldots, b)$ and the last one is $\pi'_{n_a n_b} = (b, b, \ldots, b, a, a \ldots, a)$, then the a in Position j in π'_0 needs to move to Position $n_b + j$ by exactly one transposition with each b. Each of these n_b transpositions occurs when such an a is in Position $j, j + 1, \ldots, j + n_b - 1$. Let Π'_j be the subsequence of Π' containing all permutations obtained from one of these n_b transpositions. Then $v(\Pi'_j) = (\underbrace{0, 0, \ldots, 0}_{j-1}, \underbrace{1, 1, \ldots, 1}_{n_b}, \underbrace{0, 0, \ldots, 0}_{n_a - j})$. Since Π' is the disjoint union of $\Pi'_1, \Pi'_2, \ldots, \Pi'_{n_a}$ then $v(\Pi') = \sum_{1 \leq j \leq n_a} v(\Pi'_j)$. We obtain the required equality by expanding this sum. ∎

Proof of Theorem 3.

Assume that $\mathbf{\Pi}$ is decomposed into classes A, B, and C. Then there is a permutation of $\mathbf{\Pi}$, say π_0, where the elements of A, B, and C are respectively in the first, middle, and last m positions of π_0; and another permutation π_j of $\mathbf{\Pi}$ with $1 \leq j \leq \binom{n}{2}$, where the elements of A, B, and C are respectively in the middle, first, and last m positions of π_j. Let Π be the n–halfperiod of $\mathbf{\Pi}$ with initial permutation π_0. Then Π includes π_j. We construct a sequence $\Pi' = (\pi'_0, \pi'_1, \ldots, \pi'_{3m^2})$ of permutations of the $3m$–tuple $(a, a, \ldots a, b, b, \ldots, b, c, c, \ldots, c)$ (with m elements of every class) as follows. We first make all elements within the same class, A, B, or C (in Π) indistinguishable. So we construct Π' by ignoring all permutations of Π that are obtained from the previous by a transposition of two elements in the same class. Thus, Π' consists exactly of $3m^2$ permutations of the $3m$–tuple $(a, a, \ldots a, b, b, \ldots, b, c, c, \ldots, c)$, one per each permutation of $\mathbf{\Pi}$ corresponding to a transposition between two elements in different classes. Since $\mathbf{\Pi}$ is 3–decomposable then $\pi'_{m^2} = (b, b, \ldots, b, a, a, \ldots, a, c, c, \ldots, c)$. We divide Π' into two subsequences: Π'_{AB} consisting of all permutations involving transpositions of the form ab (the first m^2 permutations of Π'), and Π'_C consisting of all permutations involving transpositions with a c (the last $2m^2$ permutations of Π'). By ignoring the $c's$ we can view Π' as a bichromatic sequence of permutations of $(\underbrace{a, a, \ldots, a}_{m}, \underbrace{b, b, \ldots, b}_{m})$ and then by Lemma 14 we have that $v(\Pi'_{AB}) = (1, 2, \ldots, m-1, m, m-1, \ldots, 2, 1, 0, 0, \ldots, 0)$. Similarly, by making the $a's$ and $b's$ indistinguishable, it follows that $v(\Pi'_C) = (1, 2, \ldots, m-1, m, \ldots, m, m-1, \ldots, 2, 1)$. Thus

$$v(\Pi') = v(\Pi'_{AB}) + v(\Pi'_C) = (2, 4, 6, \ldots, 2(m-1), 2m, 2m-1, \ldots, 3, 2, 1).$$

Now we take care of the permutations of Π obtained from tranpositions of elements in the same class. For $X \in \{A, B, C\}$ let Π_X be the subsequence of Π consisting of all permutations obtained from transpositions of two elements in X. Assume that $\pi_0 = (a_m, a_{m-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots c_m)$, where $A = \{a_m, a_{m-1}, \ldots, a_1\}$, $B = \{b_1, b_2, \ldots, b_m\}$, and $C = \{c_1, c_2, \ldots, c_m\}$. For $1 \leq i \leq m$ let Π_{a_i} be the sequence of Π consisting of all permutations obtained by transpositions of a_i with an element of $\{a_{i-1}, a_{i-2}, \ldots, a_1\}$. Note that a_m must be transposed with each of the $m-1$ elements of $\{a_{m-1}, \ldots a_2, a_1\}$ in different positions. Then $v(\Pi_{a_m}) \preceq (\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1, 1, \ldots, 1}_{m-1}, \underbrace{0, 0, \ldots, 0}_{m})$.

In general, for any $m \geq i > m/2$ it is easy to check that

$$v(\Pi_{a_i}) \preceq (\underbrace{0, 0, \ldots, 0}_{2m-i}, \underbrace{1, 1, \ldots, 1}_{2i-m-1}, \underbrace{0, 0, \ldots, 0}_{2m-i}) + (\underbrace{0, 0, \ldots, 0}_{\lfloor 3m/2 \rfloor - 1}, m-i, \underbrace{0, 0, \ldots, 0}_{3m - \lfloor 3m/2 \rfloor - 1}).$$

For all other values of i, that is, for $m/2 \geq i \geq 1$, the best we can guarantee is the trivial inequality $v(\Pi_{a_i}) \preceq (\underbrace{0, 0, \ldots, 0}_{\lfloor 3m/2 \rfloor - 1}, i-1, \underbrace{0, 0, \ldots, 0}_{3m - \lfloor 3m/2 \rfloor - 1})$.

Since Π_A is the disjoint union of $\Pi_{a_m}, \Pi_{a_{m-1}}, \ldots, \Pi_{a_1}$ then $v(\Pi_A) = \sum_{i=1}^{m} v(\Pi_{a_i}) \preceq$ $(\underbrace{0, 0, \ldots, 0}_{m}, 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor, \lceil \frac{m-1}{2} \rceil^2, \lceil \frac{m-1}{2} \rceil - 1, \ldots, 2, 1, \underbrace{0, 0, \ldots, 0}_{m})$.

It is immediately seen that the upper bound for $v(\Pi_A)$ is the same for $v(\Pi_B)$, and $v(\Pi_C)$. Therefore $v(\Pi) = v(\Pi') + v(\Pi_A) + v(\Pi_B) + v(\Pi_C) \leq v(\Pi') + 3v(\Pi_A)$.

Using the upper bound of $v(\Pi)$ provided by the upper bound of $v(\Pi')$ and $v(\Pi_A)$, and considering that for $1 \leq k < n/2$, $\chi_{\leq k}(\Pi)$ is the sum of firt and last k entries of $v(\Pi)$, the required inequality follows. ∎

By substituting the bound in Theorem 3 into Eq.(1), we obtain the following bound for the crossing number of any 3-decomposable set.

Corollary 15 *If S is a 3–descomposable generalized configuration of $n = 3r$ points in the plane, then* $\widetilde{\mathrm{cr}}(S) \geq 0.37\overline{962}\binom{n}{4} + O(n^3)$

We naturally conjecture that the underlying generalized configuration of every optimal pseudolinear drawing of K_n has a 3–decomposable circular sequence. A verification of this conjecture would immediately imply that the pseudolinear crossing number of K_n is at least $0.37\overline{962}\binom{n}{4} + O(n^3)$.

5 The usual and the pseudolinear crossing numbers of K_n are different for every $n \geq 10$: proof of Corollary 2

It is implicit in the proof of Theorem 2 in [1] that not only $\widetilde{\mathrm{cr}}(K_n) \geq \frac{1}{4}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$, but also that equality holds *only if* there is a generalized configuration S such that $E_k(S) = 3\binom{k+2}{2}$ for *every* $1 \leq k \leq \lfloor n/2 \rfloor - 1$.

If $n \geq 10$, then $n/3 < \lfloor n/2 \rfloor - 1$, and so it follows from Theorem 1 that $\widetilde{\mathrm{cr}}(K_n) > \frac{1}{4}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. The proof is complete by recalling that this last expression is an upper bound for $\mathrm{cr}(K_n)$. ∎

6 Some exact pseudolinear crossing numbers

Our aim in this section is to calculate the exact value of $\widetilde{\mathrm{cr}}(K_n)$ for $n \leq 12$ and for $n = 15$. Since these happen to coincide with the respective rectilinear crossing numbers, this yields a a computer–free calculation of the $\overline{\mathrm{cr}}(K_n)$ for these values of n.

The same techniques used below may be used to calculate the exact rectilinear and pseudolinear crossing numbers of K_6, K_7, K_8, and K_9. For the sake of brevity, we focus on the (more interesting and difficult) case $n \geq 10$.

Proposition 16 *The pseudolinear crossing numbers of* K_{10}, K_{11}, K_{12}, *and* K_{15} *are* 62, 102, 153, *and* 447, *respectively.*

Proof. Let $\widetilde{h}(S)$ denote the number of halving pseudolines of a generalized configuration S with n points. Here $n = |S|$ is an even integer. Note that $\widetilde{h}(S) = E_{(n-2)/2}(S)$. We recall from [13] (see also [9]) that $\widetilde{h}(10) = 13$, and $\widetilde{h}(12) = 18$. Now the total number of $(\leq (n-2)/2)$–pseudoedges equals the total number of transpositions in an n–halfperiod, namely $\binom{n}{2}$. Therefore $\sum_{k<\frac{n-2}{2}} E_k(S) \geq \binom{n}{2} - \widetilde{h}(S)$.

Now using Eq. (1) and $\chi_{\leq k+1}(S) = E_k(S) \geq 3\binom{k+2}{2}$, and (totally elementary) optimization arguments, we obtain, for $n = 10$, that $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ is minimized when $e_0(S) = 3$, $e_1(S) = 6$, $e_2(S) = 9$, and $e_3(S) = \binom{10}{2} - 13 - (3 + 6 + 9) = 14$, and so for such an optimal S, $\widetilde{\mathrm{cr}}(\mathcal{D}_S) \geq 62$. Therefore, $\widetilde{\mathrm{cr}}(K_{10}) \geq 62$. For $n = 11$ (note that here no halving pseudolines exist), we take into account the additional constraint (from Theorem 1) $E_3(S) > 3\binom{3+2}{2} = 30$, and we obtain that $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ is minimized when $e_0(S) = 3$, $e_1(S) = 6$, $e_2(S) = 9$, and $e_3(S) = 31 - (3+6+9) = 13$ (implying $e_4(S) = \binom{11}{2} - (3+6+9+13) = 24$)), and by Eq. (1) we obtain that for such an optimal S, $\widetilde{\mathrm{cr}}(\mathcal{D}_S) \geq 102$. Thus $\widetilde{\mathrm{cr}}(K_{11}) \geq 102$. For $n = 12$, we obtain that $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ is minimized when $e_0(S) = 3$, $e_1(S) = 6$, $e_2(S) = 9$, $e_3(S) = 12$, and $e_4(S) = \binom{12}{2} - 18 - (3+6+9+12) = 18$. This yields $\widetilde{\mathrm{cr}}(K_{12}) \geq 153$.

For $n = 15$. the general bound $E_k(S) \geq 3\binom{k+2}{2}$ suffices to show that if $E_4(S) > 45$, then $\widetilde{\mathrm{cr}}(\mathcal{D}_S) \geq 447$, as required. Thus we may assume that $E_4(S) = 45$ (which implies that $e_0 = 3, e_1 = 6, e_2 = 9, e_3 = 12$, and $e_4 = 15$). That is, the (any) 15–halfperiod associated to S is perfect. Using Proposition 12, we may pick a halfperiod with liberation sequence $a_1a_2a_3a_4a_5c_1c_2c_3c_4c_5$. It is clear that such a liberation sequence is 3–decomposable, and so, applying Theorem 3, we obtain $E_5(S) \geq 3\binom{5+2}{2} + 3\binom{5-5+2}{2} = 63 + 3 = 66$. Thus, in this case, $\widetilde{\mathrm{cr}}(\mathcal{D}_S)$ is minimized when $E_j(S) = 3(j+1)$ for every $j \leq 4$ and $e_5(S) = 66-(3+6+9+12+15) = 21$ (implying $e_6(S) = \binom{15}{2} - (3+6+9+12+15+21) = 39$), and so using Eq. (1) we obtain $\widetilde{\mathrm{cr}}(K_{15}) \geq 447$.

Finally, since there are rectilinear (and therefore pseudolinear) drawings of K_{10}, K_{11}, K_{12}, and K_{15} with $62, 102, 153$, and 447 crossings, respectively (see [3]), the upper bounds (and hence the proposition) follow. ∎

7 Open questions

We have shown that the rectilinear and the pseudolinear crossing numbers of K_n coincide for $n \leq 12$ and for $n = 15$ (we have actually proved they also coincide for $n = 13$, but the details are too lengthy and cumbersome to be included here). Inspired on this, we put forward the following.

Conjecture 17 *The pseudolinear and rectilinear crossing number of* K_n *are the same for every* n.

We recall from the proof in the previous section that $\widetilde{h}(n)$ (respectively, $h(n)$) denotes the maximum number of halving pseudolines (respectively, lines) in a generalized (respectively, geometrical) configuration of n points. We observe that in the previous section we make essential use of the exact values of $\widetilde{h}(n)$: in those calculations, as in every known crossing–optimal pseudolinear (respectively, rectilinear) drawing, the number of halving pseudolines (respectively, lines) is maximized. We conjecture that this is always the case (this is also being conjectured for rectilinear drawings in [6]).

Conjecture 18 *Suppose that S is a crossing–optimal generalized (respectively, geometrical) configuration of n points, that is, $\widetilde{\mathrm{cr}}(\mathcal{D}_S) = \widetilde{\mathrm{cr}}(K_n)$ (respectively, $\overline{\mathrm{cr}}(\mathcal{D}_S) = \overline{\mathrm{cr}}(K_n)$). If n is even, then the number of halving pseudolines (respectively, lines) in S is $\widetilde{h}(n)$ (respectively, $h(n)$).*

We are willing to go a bit further, and put forward the following slightly stronger conjecture.

Conjecture 19 *Suppose that S is a crossing–optimal generalized (respectively, geometrical) configuration of n points, that is, $\widetilde{\mathrm{cr}}(\mathcal{D}_S) = \widetilde{\mathrm{cr}}(K_n)$ (respectively, $\overline{\mathrm{cr}}(\mathcal{D}_S) = \overline{\mathrm{cr}}(K_n)$). Then, for every $k < \lfloor n/2 \rfloor$, the number of $(\leq k)$–pseudoedges (respectively, $(\leq k)$–edges) in S is smallest possible among all generalized (respectively, geometrical) configurations of n points.*

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