Compactness in the complex plane

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In our text we had the following Theorem as Proposition 1.4.18, here we have a proof of it. Recall that a set K is *compact* if every open cover of K contains a finite subcover of K. Also we use the notation $D(z,\varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$, i.e. the open disk centered at z with radius ε.

Theorem 1 *The following conditions are equivalent for a subset K of $\mathbb{C}$ (or of $\mathbb{R}$):*

(i) *K is closed and bounded.*

(ii) *Every sequence of points in K has subsequence which converges to some point in K.*

(iii) *K is compact.*

Proof. We first prove (i)⇒(ii). Suppose K is closed and bounded, consider a sequence $\{z_j\}_{j=1}^{\infty}$ in K, it is enough to show that this sequence contains a convergent subsequence, because since K is closed we already know (see proposition 1.4.8) that the limit of such sequence would be some point in K.

Since K is bounded then there is a disk centered at the origin containing K, by possibly shrinking and translating K, we can assume that K is contained in the square S_1 with vertices $0, 1, 1+i$, and i. Notice that the shrinking and translating do not change the convergence behavior of the sequence. Now, consider the following recursive procedure: Using a vertical segment and a horizontal segment divide S_1 into four squares of the same size S_{11} (with vertices $0, \frac{1}{2}, \frac{1}{2}+\frac{i}{2}, \frac{i}{2}$), S_{12} (with vertices $\frac{1}{2}, 1, 1+\frac{i}{2}, \frac{1}{2}+\frac{i}{2}$), S_{13} (with vertices $\frac{1}{2}+\frac{i}{2}, 1+\frac{i}{2}, 1+i, \frac{1}{2}+i$), and S_{14} (with vertices $\frac{i}{2}, \frac{1}{2}+\frac{i}{2}, \frac{1}{2}+i, i$), see figure 1. Since the sequence $\{z_i\}_{i=1}^{\infty}$ is infinite, there must be one of these squares with an infinite number of points. If S_{1j} is such square, pick an arbitrary point in it, call it w_1, and now only consider those points in the sequence $\{z_j\}_{j=1}^{\infty}$ with indices j larger than the index of w_1. Now, repeat the same procedure using S_{1j} instead of S_1 and proceed recursively. In this way we construct a sequence $\{w_i\}_{i=1}^{\infty}$ where the indices of the w_j's in the sequence $\{z_j\}_{j=1}^{\infty}$ are in order, and moreover each w_j is contained in a square of side $1/2^j$ and every subsequent w_k (with $k \geq j$) is also contained in the same square. Therefore

$$|w_j - w_k| < \frac{1}{2^j} \text{ for every } j \geq k$$

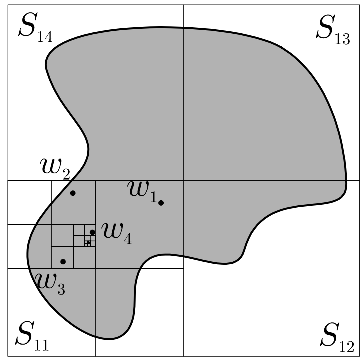

Figure 1: proof of (i) ⇒(ii)

which means that the sequence $\{w_j\}_{j=1}^{\infty}$ is a Cauchy sequence and therefore it is convergent.

Now we prove (ii)⇒(iii). Suppose that every sequence in K contains a convergent subsequence with limit in K. Consider an open cover $\mathcal{C} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ of K. We first prove the following.

Claim 2 *There is $\varepsilon > 0$ such that every $z \in K$ satisfies that there is $\alpha = \alpha(z)$ such that $D(z, \varepsilon) \subseteq U_\alpha$.*

Suppose not, then for every ε, in particular for $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$, there is a point $z_n \in K$ such that for every $\alpha \in \mathcal{A}$ we have that $D(z_n, \frac{1}{n}) \nsubseteq U_\alpha$. Consider the sequence $\{z_n\}$, by assumption there is a convergent subsequence $\{z_{n_k}\} \to z^*$. Since C is a cover we know there

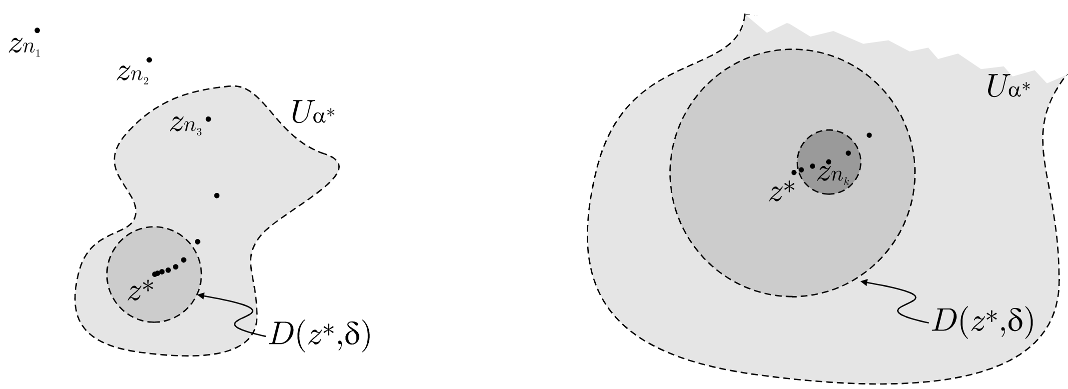

Figure 2: proof of the claim

is $\alpha^* \in \mathcal{A}$ such that $z^* \in U_{\alpha^*}$, since U_{α^*} is open there is $\delta > 0$ such that $D(z^*, \delta) \subseteq U_{\alpha^*}$. But since $\{z_{n_k}\} \to z^*$ there is N such that $|z^* - z_{n_k}| < \delta/2$ whenever $n_k \geq N$. Pick $n_k \geq N$ and also $n_k > \frac{2}{\delta}$. Notice that if $w \in D(z_{n_k}, \frac{\delta}{2})$ then

$$|z^* - w| = |z^* - z_{n_k} + z_{n_k} - w| \leq |z - z_{n_k}| + |z_{n_k} - w| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

thus $D(z_{n_k}, \frac{1}{n_k}) \subseteq D(z_{n_k}, \frac{\delta}{2}) \subseteq D(z^*, \delta) \subseteq U_{\alpha^*}$ which contradicts the fact that $D(z_n, \frac{1}{n}) \nsubseteq U_\alpha$ for every $\alpha \in \mathcal{A}$. This completes the proof of the claim.

Now, for every $z \in K$ let $D_z = D(z, \varepsilon)$ the disk given by the claim. Observe that $\{D_z\}_{z\in K}$ is an open cover of K. Suppose we know how to extract a finite subcover $\{D_{z_1}, D_{z_2}, \dots, D_{z_M}\}$ of K, then since each D_{z_j} is contained in some U_{α_j} we can deduce that $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_M}\}$ is also a finite subcover. So it is enough to show that $\{D_z\}_{z\in K}$ has a finite subcover. Let $z_1 \in K$, if D_{z_1} covers K we are done, else there is $z_2 \in K \backslash D_{z_1}$. If $\{D_{z_1}, D_{z_2}\}$ covers K we are done, else there is $z_3 \in K \backslash (D_{z_1} \cup D_{z_2})$. If we continue this process we either obtain a finite cover or a sequence $\{z_j\}$. Observe that $|z_j - z_k| > \varepsilon$ for every j and k since z_k is not in D_{z_j} and biceversa. Thus the sequence $\{z_j\}$ cannot have a convergent subsequence which is a contradiction to our main assumption. Therefore the process must end at some point and we should have a finite subcover of K.

Finally we prove (iii)⇒(i). Suppose K is compact. Consider the collection $\mathcal{C} = \{D(0;n) : n \in \mathbb{N}\}$, clearly $\mathcal{C}$ is an open cover of K (in fact of all $\mathbb{C}$), thus, since K is compact, there is a finite subcover $\mathcal{C}_F = \{D(0,n_1), D(0,n_2), \dots, D(0,n_M)\}$ of K. If we assume that $n_1 < n_2 < \cdots < n_M$ then clearly $K \subseteq D(0, n_M)$, i.e., K is bounded.

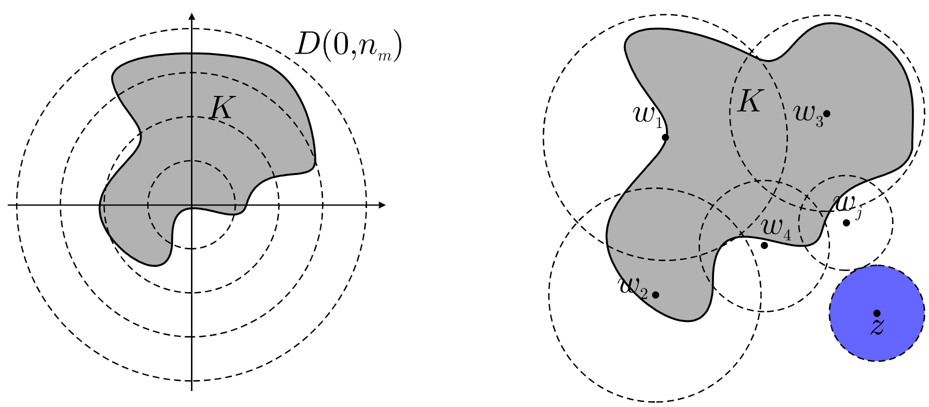

Figure 3: proof of (iii)⇒(i)

Now, to prove that K is closed we prove instead that $\mathbb{C}\backslash K$ is open. Let $z \in \mathbb{C}\backslash K$, for every $w \in K$ consider the disk $D_w = D(w, \frac{1}{2}|w - z|)$ and let $\mathcal{C} = \{D_w : w \in K\}$. Clearly $\mathcal{C}$ is an open cover of K since every point w in K has a disk centered at it in $\mathcal{C}$. Since K is compact, there is a finite subcover $\mathcal{C}_F = \{D_{w_1}, D_{w_2}, \dots, D_{w_M}\}$. Now, let $\varepsilon = \min\left\{\frac{1}{2}|w_j - z| : 1 \le j \le M\right\}$. Observe that since $\frac{1}{2}|z - w_j| > \varepsilon$ then z is farther apart than ε from every point in D_{w_j}. Indeed, if $w \in D_{w_j}$ (i.e., $|w - w_j| < \frac{1}{2}|w_j - z|$) then

$$|z - w| = |(z - w_j) + (w_j - w)| \ge |z - w_j| - |w_j - w| > |z - w_j| - \frac{1}{2}|z - w_j| = \frac{1}{2}|z - w_j| > \varepsilon.$$

Thus $D(z, \varepsilon) \cap D_{w_j} = \emptyset$ and then

$$D(z, \varepsilon) \subseteq \mathbb{C}\backslash \bigcup_{j=1}^{M} D_{w_j} \subseteq \mathbb{C}\backslash K$$

which proves that $\mathbb{C}\backslash K$ is open, i.e. K is closed. ■