Exam 2. Math 320. Fall 2010________________ Prof. Bernardo M. Ábrego

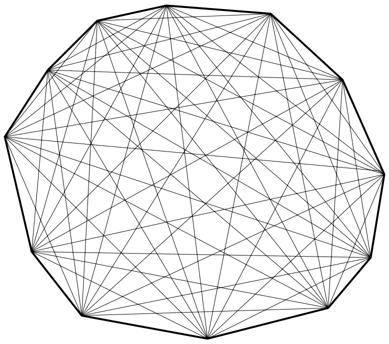
Name.____ 12/06/2010

Instructions. You have 75 minutes to complete the test. You are allowed to quote results proven in class, but you cannot quote exercises or homework problems. All the bonus parts are optional and each one is worth 5 points with no partial credit awarded (all or nothing).

1. Determine the parity **and** inverse of the permutation $f : [9] \to [9]$ with word form 594736128.

   The permutation has cycle structure (15347)(298)(6). Because it has 3 cycles and the identity permutation has 9 cycles, then the minimum number of transpositions needed to take f to the identity is 6, thus f is even. Moreover, its inverse has cycle structure (17435)(289)(6) and word form 785316492.

2. In the 12-gon shown below no three diagonals are concurrent. Count the number of pairs of diagonals that intersect inside the polygon.

   The answer is $\binom{12}{4}$. The endpoints of two diagonals that cross are the vertices of a convex quadrilateral. Reciprocally, every quadrilateral formed by 4 of the 12 vertices of the polygon is convex and thus has a pair of diagonals that intersect. Thus every set of 4 vertices of the 12-gon determines a pair of diagonals that cross and there are $\binom{12}{4}$ such 4-tuples.

3. (corrected) Let d be a common divisor of $a+b$ and $a-b$. Prove that if d is odd, then d is also a common divisor of a and b.

   Because d divides $a+b$ and $a-b$, it follows that thre are integers m and n such that $a+b = dn$ and $a-b = dm$. If we add these equations we obtain $2a = d(m+n)$. If we subtract them we get $2b = d(n-m)$. Thus d divides $2a$ and $2b$. But since d is odd, then d and 2 are relatively prime. In class we saw that if d and x are relatively prime and $d|xy$, then $d|y$. Thus $d|a$ and $d|b$.

   (a) Find the greatest common divisor of 10 379 and 1 498.

Solution 1. The number 1498 is clearly divisible by 2 and $1498 = 2 \cdot 749$. Also 749 is divisible by 7 with $749 = 7 \cdot 107$. Finally, 107 is prime because it is not divisible by 2, 3, 5, or 7. Now, 10379 is not divisible by 2 or by 7, but $10379 = 107 \cdot 97$. So the only common divisor greater than 1 is 107 and so $\gcd(10\,379, 1\,498) = 107$.

Solution 2. Using Euclid's algorithm:

$$\begin{aligned} 10379 &= 6 \cdot 1498 + 1391 \\ 1498 &= 1391 + 107 \\ 1391 &= 7 \cdot 107. \end{aligned}$$

Thus $\gcd(10\,379, 1\,498) = 107$.

(b) Find m and n such that $10\,379m + 1\,498n = \gcd(10\,379, 1\,498)$.
Backtracking part (a) we get

$$\begin{aligned} 107 &= 1498 - 1391 \Rightarrow \\ 107 &= 1498 - (10379 - 6 \cdot 1498) \Rightarrow \\ 107 &= 7 \cdot 1498 + (-1)10379. \end{aligned}$$

Thus $m = -1$ and $n = 7$ is a solution.

4. For every integer n, prove that $n^4 - n^2$ is divisible by 12. Hint: prove that the number is divisible by 3 and by 4.

Solution 1. Let $m = n^4 - n^2 = n^2(n^2 - 1) = n^2(n-1)(n+1)$. If n is even $n = 2k$ for some $k \in Z$, then $m = 4k^2(2k-1)(2k+1)$, so $4|m$. Similarly, if n is odd $n = 2k+1$for some $k \in Z$, then $m = (2k+1)^2(2k)(2k+2) = 4k(k+1)(2k+1)^2$ and so $4|m$. In both cases we conclude that $4|m$.

By the division algorithm, $n = 3k + r$ where $0 \le r \le 2$. We consider the three cases $r = 0$, $r = 1$, and $r = 2$.

If $n = 3k$ for some $k \in Z$, then $m = 9k^2(3k-1)(3k+1)$, so $3|m$. If $n = 3k+1$ for some $k \in Z$, then $m = (3k+1)^2(3k)(3k+2)$, so $3|m$. Finally, if $n = 3k+2$ for some $k \in Z$, then $m = (3k+2)^2(3k+1)(3k+3) = 3(3k+2)^2(3k+1)(k+1)$, so $3|m$.

Because $3|m$ and $4|m$ and $\gcd(3,4) = 1$, it follows that $12|m$.

Solution 2. We first prove the assertion for $n \ge 0$ by induction on n. If $n = 0$, then $n^4 - n^2 = 0$ is divisile by 12. If $n = 1$, then $n^4 - n^2 = 0$ is divisible by 12, and if $n = 2$, then $n^4 - n^2 = 16 - 4 = 12$ is clearly divisible by 12. Let $n \ge 3$, suppose by induction hypothesis that 12 divides $(n-3)^4 - (n-3)^2$. That is there is an integer k such that $(n-3)^4 - (n-3)^2 = 12k$. Expanding yields

$$\begin{aligned} 12k &= (n^4 - 12n^3 + 54n^2 - 108n + 81) - (n^2 - 6n + 9) \\ &= (n^4 - n^2) - 12n^3 + 72 + 48n^2 + 6n^2 - 108n + 6n \\ &= (n^4 - n^2) - 12(n^3 - 6 - 4n^2 + 9n) + 6n^2 + 6n \\ &= (n^4 - n^2) - 12(n^3 - 4n^2 + 9n - 6) + 12\binom{n+1}{2}. \end{aligned}$$

Therefore $n^4 - n^2 = 12(k + n^3 - 4n^2 + 9n - 6 - \binom{n+1}{2})$, and thus 12 divides $n^4 - n^2$, which completes the proof by induction. If $n < 0$, then note that $(-n)^4 - (-n)^2 = n^4 - n^2$ and we have proved already that $(-n)^4 - (-n)^2$ is divisibe by 12.

5. Count the number of solutions $(x_1, x_2, x_3, x_4, x_5)$ in nonnegative integers to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16.$$

(Extra) Count the number of solutions if the variables are positive integers.

We need to count combinations with repetitions of 16 objects taken from 5 different kinds. The answer is $\binom{16+5-1}{5-1} = \binom{20}{4}$.

For the extra part, let $x_i = 1 + y_i$ for $1 \leq i \leq 5$, where $y_i \geq 0$. Clearly the number of nonnegative integer solutions $(y_1, y_2, y_3, y_4, y_5)$ to the equation $y_1 + y_2 + y_3 + y_4 + y_5 = 11$ are in 1–1 correspondence with the number of positive integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$. Thus the answer is $\binom{11+5-1}{5-1} = \binom{15}{4}$.

6. For nonnegtive integers k, m, and n, such that $k \leq m \leq n$, prove that

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

(Extra) Use the previous part to show that for $n \geq 0$,

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

Solution 1. We count the number of committees of k people we can form from a group with m men and n women. Counting directly the answer is $\binom{m+n}{k}$. On the other hand we can condition on the number of men in the committee. Let i be the number of men in the committee, where $0 \leq i \leq k$. There are $\binom{m}{i}$ ways of selecting the i men for the committee. The remaining $k - i$ members must be women, and there are $\binom{n}{k-i}$ ways of selecting them. Thus for a fixed number i of men in the committee, there are $\binom{m}{i}\binom{n}{k-i}$ ways of selecting the committee. Thus the total number of ways is

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i},$$

which proves the identity.

Solution 2. We proceed by induction on $s = n + m + k$. If $s = 0$, then $n = m = k = 0$. The left hand side is $\binom{m}{0}\binom{n}{0} = \binom{0}{0}\binom{0}{0} = 1$ and the right hand side is $\binom{m+n}{k} = \binom{0+0}{0} = 1$, which proves the basis. Suppose that $s \geq 1$ and by induction hypothesis

$$\sum_{i=0}^{k'} \binom{m'}{i} \binom{n'}{k'-i} = \binom{m'+n'}{k'} \tag{1}$$

for every $k' \leq m' \leq n'$ such that $s' = n' + m' + k' < s$.

Now note that

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m}{0}\binom{n}{k} + \sum_{i=1}^{k} \binom{m}{i}\binom{n}{k-i},$$

by Pascal's Identity $\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$, so

$$\begin{aligned}\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} &= \binom{n}{k} + \sum_{i=1}^{k} \left(\binom{m-1}{i} + \binom{m-1}{i-1}\right)\binom{n}{k-i} \\ &= \binom{m-1}{0}\binom{n}{k} + \sum_{i=1}^{k} \binom{m-1}{i}\binom{n}{k-i} + \sum_{i=1}^{k} \binom{m-1}{i-1}\binom{n}{k-i},\end{aligned}$$

including the first term in the first sum and changing the sumation index to $j = i - 1$ in the second sum yields

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \sum_{i=0}^{k} \binom{m-1}{i}\binom{n}{k-i} + \sum_{j=0}^{k-1} \binom{m-1}{j}\binom{n}{k-1-j}.$$

Now we use the induction hypothesis for $(n', m', k') = (n, m-1, k)$ in the first sum and $(n', m', k') = (n, m-1, k-1)$ in the second sum. Note that their corresponding sums $s' = n' + m' + k'$ are equal to $s - 1$ and $s - 2$, respectively, both less than s. Therefore by Identity (1) and Pascal's Identity again,

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n-1}{k} + \binom{m+n-1}{k-1} = \binom{m+n}{k}.$$

For the extra part use $m = n = k$ and note that, as we saw in class, $\binom{n}{n-i} = \binom{n}{i}$. Thus

$$\sum_{i=0}^{k} \binom{n}{i}\binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k} = \binom{2n}{n}.$$

7. Let ab be a 2-digit natural number ($a \neq 0$) written in base 10. Prove that the 6-digit number $ababab$ has at least four distinct prime factors.

   (Extra) Find all 6-digit numbers $ababab$ with exactly four distinct prime factors.

   (Extra) Prove that all 6-digit numbers $ababab$ have at most 6 different prime factors.

   Let $n = (ababab)_{10}$. Thus

$$\begin{aligned} n &= 10^5 a + 10^4 b + 10^3 a + 10^2 b + 10a + b \\ &= (10a + b)(10^4 + 10^2 + 1).\end{aligned}$$

But $10^4 + 10^2 + 1 = 10101 = 3 \cdot 7 \cdot 13 \cdot 37$, so n has at least these 4 prime factors. For the first extra part, the 2-digit number $(ab)_{10} = 10a + b$ should only be divisible by primes in the set $\{3, 7, 13, 37\}$. The only possibilities are 13, 37, $7 \cdot 3 = 21$, $3 \cdot 13 = 39$, and $7 \cdot 13 = 91$, so the respective 6-digit numbers are 131313, 373737, 212121, 393939, and 919191.

For the second extra part, suppose that n has at least 7 different prime factors. This means that $(ab)_{10} = 10a + b$ must have at least 3 different prime factors not in the set $\{3, 7, 13, 37\}$. The three smallest primes with this property are 2, 5, and 11; bu their product is 110 which exceeds the domain of 2-digit numbers, so we get a contradiction. Therefore there are at most 6 different prime factors in n.