1. The statement below is not always true for \(x, y \in \mathbb{R}\). Give an example where it is false, and add a hypothesis on \(y\) that makes it a true statement.

“If \(x\) and \(y\) are nonzero real numbers and \(x > y\), then \((-1/x) > (-1/y)\).”

Let \(x = 1\) and \(y = -1\), they are both nonzero and \(x > y\), however

\[(-1/x) = -1 < 1 = (-1/(-1)) = (-1/y).\]

If we add the hypothesis \(y > 0\) then the statement is true: start with \(x > y\), because \(y > 0\), it follows that \(x > y > 0\). Thus \(xy > 0\) and we can divide both sides of the inequality \(x > y\) by \(xy\), to get \(1/y > 1/x\). Finally, multiplying by \((-1)\) the inequality is reversed and we get \((-1/x) > (-1/y)\).

2. Let \(f\) be a function from \(\mathbb{R}\) to \(\mathbb{R}\). Without using words of negation, write a sentence that expresses the negation of the following statement:

“For all \(b \in \mathbb{R}\), there is an \(x \in \mathbb{R}\) such that \(f(x) = b\).”

There is a \(b \in \mathbb{R}\), such that for all \(x \in \mathbb{R}\), \(f(x) \neq b\).

3. Let \(f : A \to B\) and \(g : B \to C\) be injective functions on their respective domains. Prove that \(g \circ f : A \to C\) is an injection.

Let \(h = g \circ f\) and suppose that \(h(c_1) = h(c_2)\). Thus \(g(f(c_1)) = g(f(c_2))\) and because \(g\) is injective, it follows that \(f(c_1) = f(c_2)\), but \(f\) is injective too, so it follows that \(c_1 = c_2\). Therefore \(h\) is injective.

4. Let \(P(x)\) be the assertion “\(x\) is odd”, and let \(Q(x)\) be the assertion “\(x^2 - 1\) is even”. Consider the following statements:

(a) \((\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]\).
(b) \((\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]\).

Prove that both (a) and (b) are true. Hint for part (b): Use the contrapositive.

(a). Let \(x\) be an odd number, then there is \(k \in \mathbb{Z}\), such that \(x = 2k + 1\). It follows that \(x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 2(2k^2 + 2k)\). Therefore \(x^2 - 1\) is even.

(b). We prove the contrapositive. Suppose \(\neg P(x)\) is true, that is \(x\) is not an odd integer. Then \(x\) is even and there is \(k \in \mathbb{Z}\) such that \(x = 2k\). It follows that \(x^2 - 1 = 4k^2 - 1 = 2(2k^2) - 1\) is odd, so \(\neg Q(x)\) is true.
5. Determine the set of natural numbers \( n \) for which the inequality \( 3^n > 2n^3 \) holds.

The answer is \( \{1\} \cup \{n \in \mathbb{N} : n \geq 6\} \), or \( \mathbb{N} - \{2, 3, 4, 5\} \). If \( n = 1 \), then \( 3^n = 3 > 2 = 2n^3 \). If \( n = 2 \), then \( 3^n = 9 < 16 = 2n^3 \). If \( n = 3 \), then \( 3^n = 3^3 = 27 < 54 = 2n^3 \). If \( n = 4 \), then \( 3^n = 81 < 128 = 2n^3 \). If \( n = 5 \), then \( 3^n = 3^5 = 243 < 250 = 2n^3 \). Finally, if \( n \geq 6 \) the statement is always true and we prove it by induction on \( n \). If \( n = 6 \), then \( 3^n = 729 > 532 = 2n^3 \). Suppose by induction hypothesis that \( 3^n > 2n^3 \) for some \( n \geq 6 \). Multiplying both sides by 3 we get

\[
3^{n+1} = 3 \cdot 3^n > 6n^3 = 2n^3+4n^3.
\]

because \( n \geq 6 \), then \( 4n^3 = (4n^2) \geq 6 \cdot 4n^2 = 24n^2 \). Thus

\[
3^{n+1} = 3 \cdot 3^n > 6n^3 = 2n^3+4n^3 \\
\geq 2n^3+24n^2 = 2(n^3+12n^2) = 2(n^3+3n^2+3n^2+n^2+5n^2) \\
> 2(n^3+3n^2+3n+1) = 2(n+1)^3,
\]

which proves the statement by induction. Note that \( n^2 \geq n \geq 1 \) for every positive integer \( n \).

6. Let \( \langle a \rangle \) be a recursive sequence defined by \( a_1 = 0 \), \( a_2 = 2 \), and \( a_n = 4a_{n-1} - 3a_{n-2} \) for any integer \( n \geq 3 \). Prove that \( a_n = 3^{n-1} - 1 \) for all natural numbers. Hint: Use strong induction.

If \( n = 1 \), then \( a_1 = 0 = 3^0 - 1 \). If \( n = 2 \), then \( a_2 = 2 = 3^1 - 1 \). Assume by induction hypothesis that \( n \geq 2 \) and for every \( k < n \) we have that \( a_k = 3^{k-1} - 1 \). Then by definition,

\[
a_n = 4a_{n-1} - 3a_{n-2}.
\]

Noting that \( n-1 \) and \( n-2 \) are both less than \( n \) and at least 0, we apply the induction hypothesis to \( a_{n-1} \) and \( a_{n-2} \), that is \( a_{n-1} = 3^{n-2} - 1 \) and \( a_{n-2} = 3^{n-3} - 1 \). It follows that

\[
a_n = 4(3^{n-2} - 1) - 3(3^{n-3} - 1) \\
= 4 \cdot 3^{n-2} - 4 - 3 \cdot 3^{n-3} + 3 \\
= 4 \cdot 3 \cdot 3^{n-3} - 3 \cdot 3^{n-3} - 1 \\
= 3^{n-3}(12 - 3) - 1 = 3^{n-3} \cdot 9 - 1 = 3^{n-3} \cdot 3^2 - 1 = 3^{n-1} - 1.
\]

Thus the result is true by induction.

7. A clerk returns 10 hats to 10 people who have checked them, but not necessarily in the right order. For which \( k \) is it possible that exactly \( k \) people get a wrong hat? Prove your answer.

The answer is for any \( k \), \( 2 \leq k \leq 10 \) or \( k = 0 \). First we argue that \( k = 1 \) is impossible. Indeed, if everyone except perhaps person \( p \) has the right hat, then there are 9 people
which have the right hat. This leaves only one remaining hat which has to be the hat of person \( p \), which means that \( p \) actually has the correct hat. Let \( p_1, \ldots, p_{10} \) denote the 10 people and \( h_1, \ldots, h_{10} \) their corresponding hats. To see that the other values are possible consider the following assignments for every \( k \geq 2 \): first \( p_1 \) gets \( h_k \), second, if \( 2 \leq i \leq k \), then \( p_i \) gets \( h_{i-1} \), and last, if \( k + 1 \leq i \leq 10 \), then \( p_i \) gets \( h_i \). In this assignment exactly the persons \( p_1, \ldots, p_k \) get the wrong hats. Finally, it is possible that every person receives his/her own hat and \( k = 0 \).

8. Let \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(a, b) = \frac{ab(a+b)}{2} \). Prove that \( f \) is not a surjection. Hint: look for a small natural number which is not in the image set.

Suppose \( f(a, b) = 2 \). It follows that

\[
\frac{ab(a+b)}{2} = 2
\]

for some \( a, b \in \mathbb{N} \). Thus \( ab(a+b) = 4 \). But, if \( a \) and \( b \) are at least 2, then \( a \geq 2 \), \( b \geq 2 \), and \( a + b \geq 4 \). It follows that \( ab(a+b) \geq 16 \). If \( a = 1 \) and \( b \geq 2 \), then \( a + b \geq 3 \) and thus \( ab(a+b) \geq 6 \). The only remaining pair is \((a, b) = (1, 1)\), and in that case \( ab(a+b) = 2 \). Thus there are no pairs \((a, b)\) of natural numbers such that \( f(a, b) = 2 \). Therefore \( f \) is not a surjective function.