Exam 1. Math 320. Fall 2010____________________ Prof. Bernardo M. Ábrego

Name.____

1. The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on y that makes it a true statement.

   "If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

   Let $x = 1$ and $y = -1$, they are both nonzero and $x > y$, however

   $$(-1/x) = -1 < 1 = (-1/(-1)) = (-1/y).$$

   If we add the hypothesis $y > 0$ then the statement is true: start with $x > y$, because $y > 0$, it follows that $x > y > 0$. Thus $xy > 0$ and we can divide both sides of the inequality $x > y$ by xy, to get $1/y > 1/x$. Finally, multiplying by (-1) the inequality is reversed and we get $(-1/x) > (-1/y)$.

2. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Without using words of negation, write a sentence that expresses the negation of the following statement:

   "For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$."

   There is a $b \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, $f(x) \neq b$.

3. Let $f : A \to B$ and $g : B \to C$ be injective functions on their respective domains. Prove that $g \circ f : A \to C$ is an injection.

   Let $h = g \circ f$ and suppose that $h(c_1) = h(c_2)$. Thus $g(f(c_1)) = g(f(c_2))$ and because g is injective, it follows that $f(c_1) = f(c_2)$, but f is injective too, so it follows that $c_1 = c_2$. Therefore h is injective.

4. Let $P(x)$ be the asssertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is even". Consider the following statements:

   (a) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$.

   (b) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

   Prove that both (a) and (b) are true. Hint for part (b): Use the contrapositive.

   (a). Let x be an odd number, then there is $k \in \mathbb{Z}$, such that $x = 2k + 1$. It follows that $x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 2(2k^2 + 2k)$. Therefore $x^2 - 1$ is even.

   (b). We prove the contrapositive. Suppose $\neg P(x)$ is true, that is x is not an odd integer. Then x is even and there is $k \in \mathbb{Z}$ such that $x = 2k$. It follows that $x^2 - 1 = 4k^2 - 1 = 2(2k^2) - 1$ is odd, so $\neg Q(x)$ is true.

5. Determine the set of natural numbers n for which the inequality $3^n > 2n^3$ holds.

The answer is $\{1\} \cup \{n \in \mathbb{N} : n \geq 6\}$, or $\mathbb{N} - \{2, 3, 4, 5\}$. If $n = 1$, then $3^n = 3 > 2 = 2n^3$. If $n = 2$, then $3^n = 9 < 16 = 2n^3$. If $n = 3$, then $3^n = 3^3 = 27 < 54 = 2n^3$. If $n = 4$, then $3^n = 81 < 128 = 2n^3$. If $n = 5$, then $3^n = 3^5 = 243 < 250 = 2n^3$. Finally, if $n \geq 6$ the statement is always true and we prove it by induction on n. If $n = 6$, then $3^n = 729 > 532 = 2n^3$. Suppose by induction hypothesis that $3^n > 2n^3$ for some $n \geq 6$. Multiplying both sides by 3 we get

$$3^{n+1} = 3 \cdot 3^n > 6n^3 = 2n^3 + 4n^3.$$

because $n \geq 6$, then $4n^3 = n\left(4n^2\right) \geq 6 \cdot 4n^2 = 24n^2$. Thus

$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n > 6n^3 = 2n^3 + 4n^3 \\ &\geq 2n^3 + 24n^2 = 2\left(n^3 + 12n^2\right) = 2\left(n^3 + 3n^2 + 3n^2 + n^2 + 5n^2\right) \\ &> 2(n^3 + 3n^2 + 3n + 1) = 2(n+1)^3, \end{aligned}$$

which proves the statement by induction. Note that $n^2 \geq n \geq 1$ for every positive integer n.

6. Let $\langle a \rangle$ be a recursive sequence defined by $a_1 = 0$, $a_2 = 2$, and $a_n = 4a_{n-1} - 3a_{n-2}$ for any integer $n \geq 3$. Prove that $a_n = 3^{n-1} - 1$ for all natural numbers. Hint: Use strong induction.

If $n = 1$, then $a_1 = 0 = 3^0 - 1$. If $n = 2$, then $a_2 = 2 = 3^1 - 1$. Assume by induction hypothesis that $n \geq 2$ and for every $k < n$ we have that $a_k = 3^{k-1} - 1$. Then by definition,

$$a_n = 4a_{n-1} - 3a_{n-2}.$$

Noting that $n-1$ and $n-2$ are both less than n and at least 0, we apply the induction hypothesis to a_{n-1} and a_{n-2}, that is $a_{n-1} = 3^{n-2} - 1$ and $a_{n-2} = 3^{n-3} - 1$. It follows that

$$\begin{aligned} a_n &= 4(3^{n-2} - 1) - 3(3^{n-3} - 1) \\ &= 4 \cdot 3^{n-2} - 4 - 3 \cdot 3^{n-3} + 3 \\ &= 4 \cdot 3 \cdot 3^{n-3} - 3 \cdot 3^{n-3} - 1 \\ &= 3^{n-3}(12 - 3) - 1 = 3^{n-3} \cdot 9 - 1 = 3^{n-3} \cdot 3^2 - 1 = 3^{n-1} - 1. \end{aligned}$$

Thus the result is true by induction.

7. A clerk returns 10 hats to 10 people who have checked them, but not necessarily in the right order. For which k is it possible that exactly k people get a wrong hat? Prove your answer.

The answer is for any k, $2 \leq k \leq 10$ or $k = 0$. First we argue that $k = 1$ is impossible. Indeed, if everyone except perhaps person p has the right hat, then there are 9 people which have the right hat. This leaves only one remaining hat which has to be the hat of person p, which means that p actually has the correct hat. Let $p_1, \ldots, p_{10}$ denote the 10 people and $h_1, \ldots, h_{10}$ their corresponding hats. To see that the other values are possible consider the following assignments for every $k \geq 2$: first p_1 gets h_k, second, if $2 \leq i \leq k$, then p_i gets h_{i-1}, and last, if $k+1 \leq i \leq 10$, then p_i gets h_i. In this assignment exactly the persons $p_1, \ldots, p_k$ get the wrong hats. Finally, it is possible that every person receives his/her own hat and $k = 0$.

8. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(a, b) = ab(a+b)/2$. Prove that f is not a surjection. Hint: look for a small natural number which is not in the image set.

   Suppose $f(a, b) = 2$. It follows that

$$\frac{ab(a+b)}{2} = 2$$

   for some $a, b \in \mathbb{N}$. Thus $ab(a+b) = 4$. But, if a and b are at least 2, then $a \geq 2$, $b \geq 2$, and $a + b \geq 4$. It follows that $ab(a+b) \geq 16$. If $a = 1$ and $b \geq 2$, then $a + b \geq 3$ and thus $ab(a+b) \geq 6$. The only remaining pair is $(a, b) = (1, 1)$, and in that case $ab(a+b) = 2$. Thus there are no pairs (a, b) of natural numbers such that $f(a, b) = 2$. Therefore f is not a surjective function.