Correlation and Regression

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Major Points - Correlation

- Questions answered by correlation
- Scatterplots
- An example
- The correlation coefficient
- Other kinds of correlations
- Factors affecting correlations
- Testing for significance

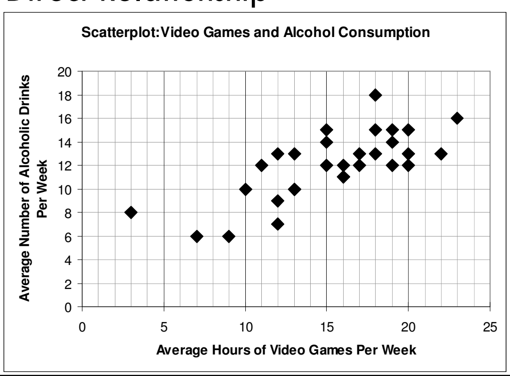
The Question

- Are two variables related?
  - Does one increase as the other increases?
    - e. g. skills and income
  - Does one decrease as the other increases?
    - e. g. health problems and nutrition
- How can we get a numerical measure of the degree of relationship?

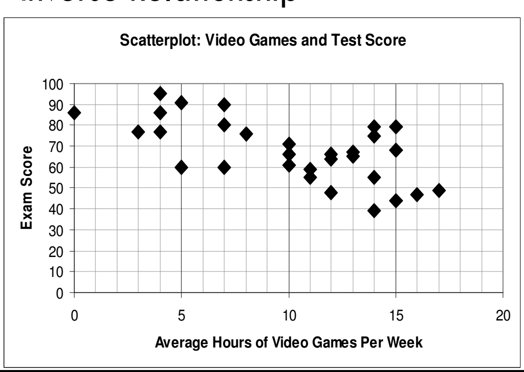
Scatterplots

- AKA scatter diagram or scattergram.
- Graphically depicts the relationship between two variables in two dimensional space.

Direct Relationship



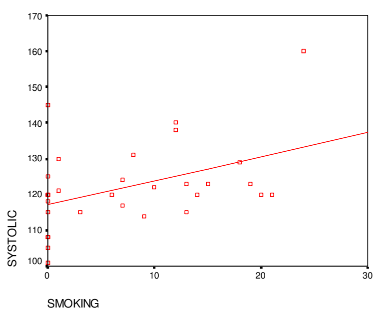
Inverse Relationship



An Example

- Does smoking cigarettes increase systolic blood pressure?
- Plotting number of cigarettes smoked per day against systolic blood pressure
  - Fairly moderate relationship
  - Relationship is positive

Trend?



Smoking and BP

- Note relationship is moderate, but real.
- Why do we care about relationship?
  - What would conclude if there were no relationship?
  - What if the relationship were near perfect?
  - What if the relationship were negative?

Heart Disease and Cigarettes

- Data on heart disease and cigarette smoking in 21 developed countries (Landwehr and Watkins, 1987)
- Data have been rounded for computational convenience.
  - The results were not affected.

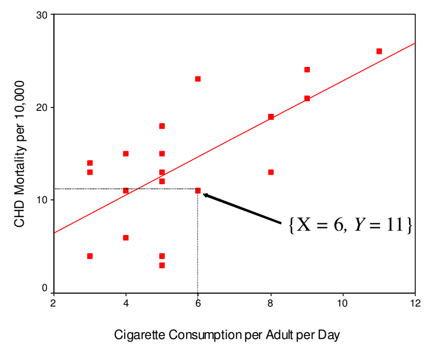
The Data

Surprisingly, the U.S. is the first country on the list--the country with the highest consumption and highest mortality.

Country	Cigarettes	CHD
1	11	26
2	9	21
3	9	24
4	9	21
5	8	19
6	8	13
7	8	19
8	6	11
9	6	23
10	5	15
11	5	13
12	5	4
13	5	18
14	5	12
15	5	3
16	4	11
17	4	15
18	4	6
19	3	13
20	3	4
21	3	14

Scatterplot of Heart Disease

- CHD Mortality goes on ordinate (Y axis)
  - Why?
- Cigarette consumption on abscissa (X axis)
  - Why?
- What does each dot represent?
- Best fitting line included for clarity



What Does the Scatterplot Show?

- As smoking increases, so does coronary heart disease mortality.
- Relationship looks strong
- Not all data points on line.
  - This gives us "residuals" or "errors of prediction"
    - To be discussed later

Correlation

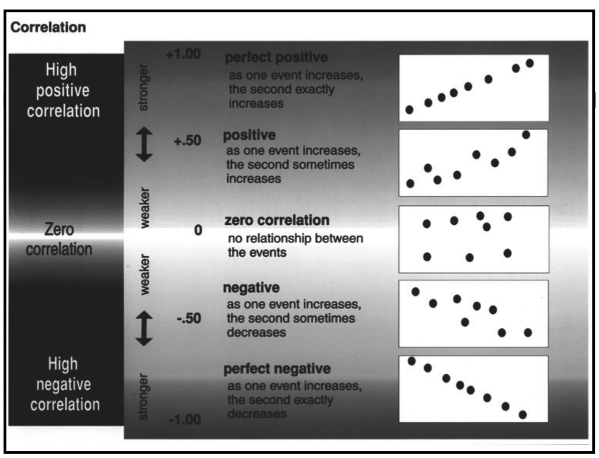
- Co-relation
- The relationship between two variables
- Measured with a correlation coefficient
- Most popularly seen correlation coefficient: Pearson Product-Moment Correlation

Types of Correlation

- Positive correlation
  - High values of X tend to be associated with high values of Y.
  - As X increases, Y increases
- Negative correlation
  - High values of X tend to be associated with low values of Y.
  - As X increases, Y decreases
- No correlation
- No consistent tendency for values on Y to increase or decrease as X increases

Correlation Coefficient

- A measure of degree of relationship.
- Between 1 and -1
- Sign refers to direction.
- Based on covariance
  - Measure of degree to which large scores on X go with large scores on Y, and small scores on X go with small scores on Y
  - Think of it as variance, but with 2 variables instead of 1 (What does that mean??)



Covariance

- Remember that variance is:

$$Var_X = \frac{\Sigma(X-\overline{X})^2}{N-1} = \frac{\Sigma(X-\overline{X})(X-\overline{X})}{N-1}$$

- The formula for co-variance is:

$$Cov_{XY} = \frac{\Sigma(X-\overline{X})(Y-\overline{Y})}{N-1}$$

- How this works, and why?
- When would cov_{XY} be large and positive? Large and negative?

Example

Country	X (Cig.)	Y (CHD)	$(X-\overline{X})$	$(Y-\overline{Y})$	$(X-\overline{X})*(Y-\overline{Y})$
1	11	26	5.05	11.48	57.97
2	9	21	3.05	6.48	19.76
3	9	24	3.05	9.48	28.91
4	9	21	3.05	6.48	19.76
5	8	19	2.05	4.48	9.18
6	8	13	2.05	-1.52	-3.12
7	8	19	2.05	4.48	9.18
8	6	11	0.05	-3.52	-0.18
9	6	23	0.05	8.48	0.42
10	5	15	-0.95	0.48	-0.46
11	5	13	-0.95	-1.52	1.44
12	5	4	-0.95	-10.52	9.99
13	5	18	-0.95	3.48	-3.31
14	5	12	-0.95	-2.52	2.39
15	5	3	-0.95	-11.52	10.94
16	4	11	-1.95	-3.52	6.86
17	4	15	-1.95	0.48	-0.94
18	4	6	-1.95	-8.52	16.61
19	3	13	-2.95	-1.52	4.48
20	3	4	-2.95	-10.52	31.03
21	3	14	-2.95	-0.52	1.53
Mean	5.95	14.52			
SD	2.33	6.69			
Sum					222.44

Example

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$$Cov_{cig.\&CHD} = \frac{\Sigma(X-\overline{X})(Y-\overline{Y})}{N-1} = \frac{222.44}{21-1} = 11.12$$

- What the heck is a covariance?
- I thought we were talking about correlation?

Correlation Coefficient

- Pearson's Product Moment Correlation
- Symbolized by *r*
- Covariance ÷ (product of the 2 SDs)

$$r = \frac{Cov_{XY}}{s_X s_Y}$$

- Correlation is a standardized covariance

Calculation for Example

- $Cov_{XY} = 11.12$
- $s_X = 2.33$
- $s_Y = 6.69$

$$r = \frac{\text{cov}_{XY}}{s_X s_Y} = \frac{11.12}{(2.33)(6.69)} = \frac{11.12}{15.59} = .713$$

Example

- Correlation = .713
- Sign is positive
  - Why?
- If sign were negative
  - What would it mean?
  - Would not alter the *degree* of relationship.

Other calculations

- Z-score method

$$r = \frac{\sum z_x z_y}{N-1}$$

- Computational (Raw Score) Method

$$r = \frac{N\sum XY - \sum X \sum Y}{\sqrt{\left[N\sum X^2 - (\sum X)^2\right]\left[N\sum Y^2 - (\sum Y)^2\right]}}$$

Other Kinds of Correlation

- Spearman Rank-Order Correlation Coefficient (r_{sp})
  - used with 2 ranked/ordinal variables
  - uses the same Pearson formula

Attractiveness	Symmetry
3	2
4	6
1	1
2	3
5	4
6	5

r_{sp} = 0.77

Other Kinds of Correlation

- Point biserial correlation coefficient (r_{pb})
  - used with one continuous scale and one nominal or ordinal or dichotomous scale.
  - uses the same Pearson formula

Attractiveness	Date?
3	0
4	0
1	1
2	1
5	1
6	0

r_{pb} = -0.49

Other Kinds of Correlation

- Phi coefficient (Φ)
  - used with two dichotomous scales.
  - uses the same Pearson formula

Attractiveness	Date?
0	0
1	0
1	1
1	1
0	0
1	1

$\Phi = 0.71$

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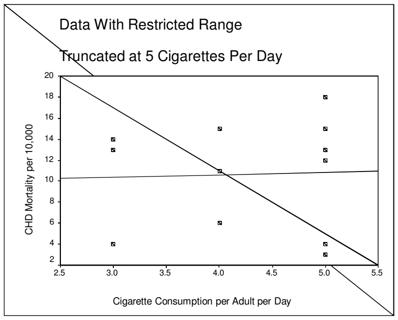
Factors Affecting *r*

- Range restrictions
  - Looking at only a small portion of the total scatter plot (looking at a smaller portion of the scores' variability) **decreases** *r*.
  - Reducing variability reduces *r*
- Nonlinearity
  - The Pearson r (and its relatives) measure the degree of **linear** relationship between two variables
  - If a strong non-linear relationship exists, r will provide a low, or at least inaccurate measure of the true relationship.

Factors Affecting *r*

- Heterogeneous subsamples
  - Everyday examples (e.g. height and weight using both men and women)
- Outliers
  - Overestimate Correlation
  - Underestimate Correlation

Countries With Low Consumptions



Truncation

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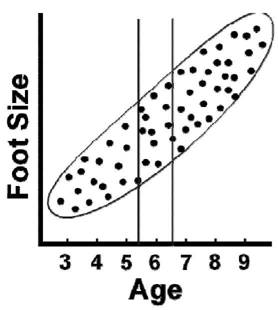

Non-linearity

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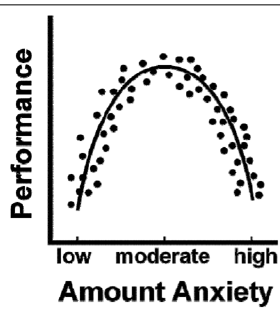

Heterogenous samples

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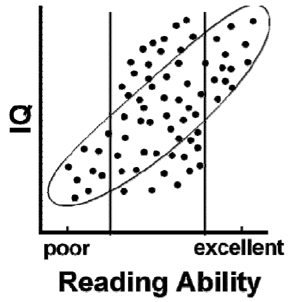

Outliers

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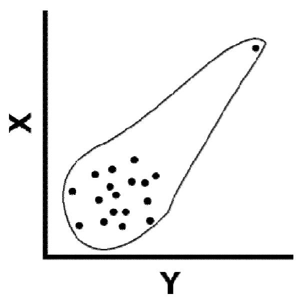

Testing Correlations

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- So you have a correlation. Now what?
- In terms of magnitude, how big is big?
  - Small correlations in large samples are "big."
  - Large correlations in small samples aren't always "big."
- Depends upon the magnitude of the correlation coefficient

AND

- The size of your sample.

Testing r

- Population parameter = ρ
- Null hypothesis H_0: $\rho = 0$
  - Test of linear independence
  - What would a true null mean here?
  - What would a false null mean here?
- Alternative hypothesis (H_1) $\rho \neq 0$
  - Two-tailed

Tables of Significance

- We can convert r to t and test for significance:

$$t = r\sqrt{\frac{N-2}{1-r^2}}$$

- Where DF = N-2

Tables of Significance

- In our example r was .71
- N-2 = 21 – 2 = 19

$$t = r\sqrt{\frac{N-2}{1-r^2}} = .71 * \sqrt{\frac{19}{1-.71^2}} = .71 * \sqrt{\frac{19}{.4959}} = 6.90$$

- T-crit (19) = 2.09
- Since 6.90 is larger than 2.09 reject $\rho = 0$.

Computer Printout

- Printout gives test of significance.

Correlations

		CIGARET	CHD
CIGARET	Pearson Correlation	1	.713**
	Sig. (2-tailed)	.	.000
	N	21	21
CHD	Pearson Correlation	.713**	1
	Sig. (2-tailed)	.000	.
	N	21	21

**. Correlation is significant at the 0.01 level (2-tailed).

Regression

What is regression?

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- How do we predict one variable from another?
- How does one variable change as the other changes?
- Influence

Linear Regression

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- A technique we use to predict the most likely score on one variable from those on another variable
- Uses the *nature of the relationship* (i.e. correlation) between two variables to *enhance* your prediction

Linear Regression: Parts

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- Y - the variables you are predicting
  - i.e. dependent variable
- X - the variables you are using to predict
  - i.e. independent variable
- $\hat{Y}$- your predictions (also known as Y')

Why Do We Care?

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- We may want to make a prediction.
- More likely, we want to understand the relationship.
  - How fast does CHD mortality rise with a one unit increase in smoking?
  - Note: we speak about predicting, but often don't actually predict.

An Example

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- Cigarettes and CHD Mortality again
- Data repeated on next slide
- We want to predict level of CHD mortality in a country averaging 10 cigarettes per day.

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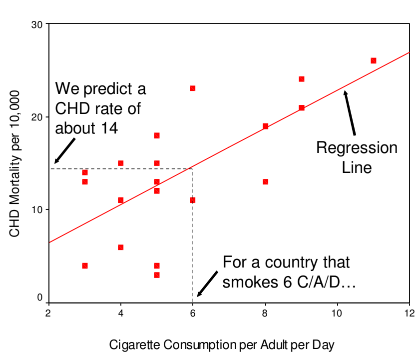
The Data

Based on the data we have what would we predict the rate of CHD be in a country that smoked 10 cigarettes on average?
First, we need to establish a prediction of CHD from smoking…

Country	Cigarettes	CHD
1	11	26
2	9	21
3	9	24
4	9	21
5	8	19
6	8	13
7	8	19
8	6	11
9	6	23
10	5	15
11	5	13
12	5	4
13	5	18
14	5	12
15	5	3
16	4	11
17	4	15
18	4	6
19	3	13
20	3	4
21	3	14



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Regression Line

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- Formula

$$\hat{Y} = bX + a$$

  - $\hat{Y}$= the predicted value of *Y* (e.g. CHD mortality)
  - X = the predictor variable (e.g. average cig./adult/country)

Regression Coefficients

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- "Coefficients" are *a* and *b*
- *b* = slope
  - Change in predicted *Y* for one unit change in *X*
- *a* = intercept
  - value of $\hat{Y}$ when *X* = 0

Calculation

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- Slope $b = \frac{\text{cov}_{XY}}{s_X^2}$ or $b = r\left[\frac{s_y}{s_x}\right]$

$$\text{or } b = \frac{N\sum XY - \sum X \sum Y}{\left[N\sum X^2 - (\sum X)^2\right]}$$

- Intercept $a = \overline{Y} - b\overline{X}$

For Our Data

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- Cov_{XY} = 11.12
- $s^2_X = 2.33^2 = 5.447$
- $b = 11.12/5.447 = 2.042$
- $a = 14.524 - 2.042*5.952 = 2.32$
- See SPSS printout on next slide

Answers are not exact due to rounding error and desire to match SPSS.

SPSS Printout

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Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	2.367	2.941		.805	.431
	Cigarette Consumption per Adult per Day	2.042	.461	.713	4.426	.000

a. Dependent Variable: CHD Mortality per 10,000

Note:

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- The values we obtained are shown on printout.
- The intercept is the value in the *B* column labeled "constant"
- The slope is the value in the *B* column labeled by name of predictor variable.

Making a Prediction

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- Second, once we know the relationship we can predict

$$\hat{Y} = bX + a = 2.042X + 2.367$$

$$\hat{Y} = 2.042*10 + 2.367 = 22.787$$

- We predict 22.77 people/10,000 in a country with an average of 10 C/A/D will die of CHD

Accuracy of Prediction

- Finnish smokers smoke 6 C/A/D
- We predict:

$$\hat{Y} = bX + a = 2.042X + 2.367$$

$$\hat{Y} = 2.042*6 + 2.367 = 14.619$$

- They actually have 23 deaths/10,000
- Our error (“residual”) = 23 - 14.619 = 8.38
  - a large error

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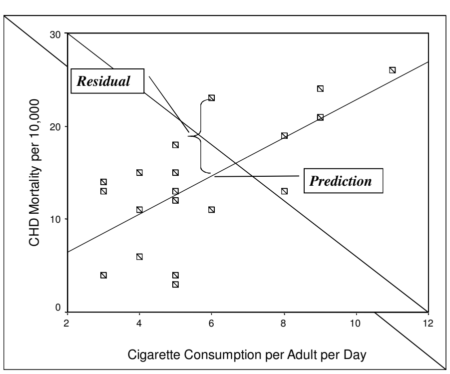

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Residuals

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- When we predict Ŷ for a given X, we will sometimes be in error.
- Y – Ŷ for any X is a an ***error of estimate***
- Also known as: a ***residual***
- We want to Σ(Y- Ŷ) as small as possible.
- BUT, there are infinitely many lines that can do this.
- Just draw ANY line that goes through the mean of the X and Y values.
- Minimize Errors of Estimate… How?

Minimizing Residuals

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- Again, the problem lies with this definition of the mean:

$$\sum (X - \bar{X}) = 0$$

- So, how do we get rid of the 0's?
- Square them.

Regression Line:
A Mathematical Definition

- The regression line is the line which when drawn through your data set produces the smallest value of:

$$\sum (Y - \hat{Y})^2$$

- Called the Sum of Squared Residual or $SS_{residual}$
- Regression line is also called a "least squares line."

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Summarizing Errors of Prediction

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- Residual variance
  - The variability of predicted values

$$s^2_{Y-\hat{Y}} = \frac{\Sigma(Y_i - \hat{Y}_i)^2}{N-2} = \frac{SS_{residual}}{N-2}$$

Standard Error of Estimate

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- Standard error of estimate
  - The standard deviation of predicted values

$$s_{Y-\hat{Y}} = \sqrt{\frac{\Sigma(Y_i - \hat{Y}_i)^2}{N-2}} = \sqrt{\frac{SS_{residual}}{N-2}}$$

- A common measure of the accuracy of our predictions
  - We want it to be as small as possible.

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Country	X (Cig.)	Y (CHD)	Y'	(Y - Y')	$(Y - Y')^2$
1	11	26	24.829	1.171	1.371
2	9	21	20.745	0.255	0.065
3	9	24	20.745	3.255	10.595
4	9	21	20.745	0.255	0.065
5	8	19	18.703	0.297	0.088
6	8	13	18.703	-5.703	32.524
7	8	19	18.703	0.297	0.088
8	6	11	14.619	-3.619	13.097
9	6	23	14.619	8.381	70.241
10	5	15	12.577	2.423	5.871
11	5	13	12.577	0.423	0.179
12	5	4	12.577	-8.577	73.565
13	5	18	12.577	5.423	29.409
14	5	12	12.577	-0.577	0.333
15	5	3	12.577	-9.577	91.719
16	4	11	10.535	0.465	0.216
17	4	15	10.535	4.465	19.936
18	4	6	10.535	-4.535	20.566
19	3	13	8.493	4.507	20.313
20	3	4	8.493	-4.493	20.187
21	3	14	8.493	5.507	30.327
Mean	5.952	14.524			
SD	2.334	6.690			
Sum				0.04	440.757

Example

$$s^2_{Y-\hat{Y}} = \frac{\Sigma(Y_i - \hat{Y}_i)^2}{N-2} = \frac{440.756}{21-2} = 23.198$$

$$s_{Y-\hat{Y}} = \sqrt{\frac{\Sigma(Y_i - \hat{Y}_i)^2}{N-2}} = \sqrt{\frac{440.756}{21-2}} =$$

$$= \sqrt{23.198} = 4.816$$

Regression and Z Scores

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- When your data are standardized (linearly transformed to z-scores), the slope of the regression line is called β
- DO NOT confuse this β with the β associated with type II errors. They're different.
- When we have one predictor, r = β
- $Z_y = \beta Z_x$, since A now equals 0

Partitioning Variability

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- Sums of square deviations
  - Total $SS_{total} = \sum (Y - \overline{Y})^2$
  - Regression $SS_{regression} = \sum (\hat{Y} - \overline{Y})^2$
  - Residual we already covered

$$SS_{residual} = \sum (Y - \hat{Y})^2$$

- $SS_{total} = SS_{regression} + SS_{residual}$

Partitioning Variability

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- Degrees of freedom
  - Total
    - $df_{total} = N - 1$
  - Regression
    - $df_{regression}$ = number of predictors
  - Residual
    - $df_{residual} = df_{total} – df_{regression}$
- $df_{total} = df_{regression} + df_{residual}$

Partitioning Variability

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- Variance (or Mean Square)
  - Total Variance
    - $s^2_{total} = SS_{total} / df_{total}$
  - Regression Variance
    - $s^2_{regression} = SS_{regression} / df_{regression}$
  - Residual Variance
    - $s^2_{residual} = SS_{residual} / df_{residual}$

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Example

Country	X (Cig.)	Y (CHD)	Y'	(Y - Y')	$(Y - Y')^2$	(Y' - Ybar)	(Y - Ybar)
1	11	26	24.829	1.171	1.371	106.193	131.699
2	9	21	20.745	0.255	0.065	38.701	41.939
3	9	24	20.745	3.255	10.595	38.701	89.795
4	9	21	20.745	0.255	0.065	38.701	41.939
5	8	19	18.703	0.297	0.088	17.464	20.035
6	8	13	18.703	-5.703	32.524	17.464	2.323
7	8	19	18.703	0.297	0.088	17.464	20.035
8	6	11	14.619	-3.619	13.097	0.009	12.419
9	6	23	14.619	8.381	70.241	0.009	71.843
10	5	15	12.577	2.423	5.871	3.791	0.227
11	5	13	12.577	0.423	0.179	3.791	2.323
12	5	4	12.577	-8.577	73.565	3.791	110.755
13	5	18	12.577	5.423	29.409	3.791	12.083
14	5	12	12.577	-0.577	0.333	3.791	6.371
15	5	3	12.577	-9.577	91.719	3.791	132.803
16	4	11	10.535	0.465	0.216	15.912	12.419
17	4	15	10.535	4.465	19.936	15.912	0.227
18	4	6	10.535	-4.535	20.566	15.912	72.659
19	3	13	8.493	4.507	20.313	36.373	2.323
20	3	4	8.493	-4.493	20.187	36.373	110.755
21	3	14	8.493	5.507	30.327	36.373	0.275
Mean	5.952	14.524					
SD	2.334	6.690					
Sum				0.04	440.757	454.307	895.247

Y' = (2.04*X) + 2.37

Example

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$$SS_{Total} = \sum (Y - \bar{Y})^2 = 895.247;\ df_{total} = 21 - 1 = 20$$

$$SS_{regression} = \sum (\hat{Y} - \bar{Y})^2 = 454.307;\ df_{regression} = 1 \text{ (only 1 predictor)}$$

$$SS_{residual} = \sum (Y - \hat{Y})^2 = 440.757;\ df_{residual} = 20 - 1 = 19$$

$$s^2_{total} = \frac{\sum (Y - \bar{Y})^2}{N - 1} = \frac{895.247}{20} = 44.762$$

$$s^2_{regression} = \frac{\sum (\hat{Y} - \bar{Y})^2}{1} = \frac{454.307}{1} = 454.307$$

$$s^2_{residual} = \frac{\sum (Y - \hat{Y})^2}{N - 2} = \frac{440.757}{19} = 23.198$$

$$Note: \sqrt{s^2_{residual}} = s_{Y - \hat{Y}}$$

Coefficient of Determination

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- It is a measure of the percent of predictable variability

  $r^2 =$ the correlation squared

  or

  $$r^2 = \frac{SS_{regression}}{SS_Y}$$

- The percentage of the total variability in Y explained by X

r^2 for our example

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- $r = .713$
- $r^2 = .713^2 = .508$
- or $$r^2 = \frac{SS_{regression}}{SS_Y} = \frac{454.307}{895.247} = .507$$
- Approximately 50% in variability of incidence of CHD mortality is associated with variability in smoking.

Coefficient of Alienation

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- It is defined as 1 - r^2 or

  $$1 - r^2 = \frac{SS_{residual}}{SS_Y}$$

- Example

  1 - .508 = .492

  $$1 - r^2 = \frac{SS_{residual}}{SS_Y} = \frac{440.757}{895.247} = .492$$

r^2, SS and $s_{Y-Y'}$

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- $r^2 * SS_{total} = SS_{regression}$
- $(1 - r^2) * SS_{total} = SS_{residual}$
- We can also use r^2 to calculate the standard error of estimate as:

$$s_{Y-\hat{Y}} = s_y\sqrt{(1-r^2)\left(\frac{N-1}{N-2}\right)} = 6.690*\sqrt{(.492)\left(\frac{20}{19}\right)} = 4.816$$

Testing Overall Model

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- We can test for the overall prediction of the model by forming the ratio:

$$\frac{s^2_{regression}}{s^2_{residual}} = F \text{ statistic}$$

- If the calculated F value is larger than a tabled value (F-Table) we have a significant prediction

Testing Overall Model

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- Example $\frac{s^2_{regression}}{s^2_{residual}} = \frac{454.307}{23.198} = 19.594$
- F-Table – F critical is found using 2 things $df_{regression}$ (numerator) and $df_{residual.}$(demoninator)
- F-Table our F_{crit} (1,19) = 4.38
- 19.594 > 4.38, significant overall
- Should all sound familiar…

SPSS output

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Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.713[a]	.508	.482	4.81640

a. Predictors: (Constant), CIGARETT

ANOVA[b]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	454.482	1	454.482	19.592	.000[a]
	Residual	440.757	19	23.198		
	Total	895.238	20			

a. Predictors: (Constant), CIGARETT

b. Dependent Variable: CHD

Testing Slope and Intercept

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- The regression coefficients can be tested for significance
- Each coefficient divided by it's standard error equals a t value that can also be looked up in a t-table
- Each coefficient is tested against 0

Testing the Slope

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- With only 1 predictor, the standard error for the slope is:

$$se_b = \frac{s_{Y-\hat{Y}}}{s_X \sqrt{N-1}}$$

- For our Example:

$$se_b = \frac{4.816}{2.334\sqrt{21-1}} = \frac{4.816}{10.438} = .461$$

Testing Slope and Intercept

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- These are given in computer printout as a *t* test.

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	2.367	2.941		.805	.431
	Cigarette Consumption per Adult per Day	2.042	.461	.713	4.426	.000

a. Dependent Variable: CHD Mortality per 10,000

Testing

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- The *t* values in the second from right column are tests on slope and intercept.
- The associated *p* values are next to them.
- The slope is significantly different from zero, but not the intercept.
- Why do we care?

Testing

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- What does it mean if slope is not significant?
  - How does that relate to test on *r*?
- What if the intercept is not significant?
- Does significant slope mean we predict quite well?