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DETERMINANTAL VARIETIES OVER TRUNCATED POLYNOMIAL RINGS

TOMAŽ KOŠIR AND B.A. SETHURAMAN

ABSTRACT. We study components and dimensions of higher order determinantal varieties obtained by considering generic $m \times n$ ($m \leq n$) matrices over rings of the form $F[t]/(t^k)$, and for some fixed r, setting the coefficients of powers of t of all $r \times r$ minors to zero. These varieties can be interpreted as spaces of $(k-1)$-th order jets over the classical determinantal varieties; a special case of these varieties first appeared in a problem in commuting matrices. We show that when $r = m$, the varieties are irreducible, but when $r < m$, these varieties are reducible. We show that when $r = 2 < m$ (any k), there are exactly $\lfloor k/2 \rfloor + 1$ components, which we determine explicitly, and for general $r < m$, we show there are at least $\lfloor k/2 \rfloor + 1$ components. We also determine the components explicitly for $k = 2$ and $k = 3$ for all values of r (for $k = 3$ for all but finitely many pairs of (m, n)).

1. INTRODUCTION

Let F be an algebraically closed field and $\mathbf{A}_F^k$ the affine space of dimension k over F. By a *variety* in $\mathbf{A}_F^k$ we will mean the zero set of a collection of polynomials over F in k variables; in particular, our varieties are not assumed irreducible. The varieties $\mathcal{Z}_r^{m,n} \subset \mathbf{A}_F^{mn}$ consisting of $m \times n$ matrices ($m \leq n$) with entries in F and of rank at most $r-1$ are of course a natural and very well understood class of objects; their various geometric and algebraic properties and their connections to representation theory and combinatorics have been extensively studied (see [2] for instance). By contrast, very little is known about the following class of objects $\mathcal{Z}_{r,k}^{m,n}$ that are very closely related to the classical varieties $\mathcal{Z}_r^{m,n}$: Consider the truncated polynomial ring $R = F[t]/(t^k)$ ($k = 1, 2, 3, \dots$), and let $X(t)$ be the generic $m \times n$ matrix over this ring; thus, each entry of X is of the form $x_{i,j}(t) = x_{i,j}^{(0)} + x_{i,j}^{(1)} t + \cdots + x_{i,j}^{(k-1)} t^{k-1}$. Each $r \times r$ minor of this matrix is an element of $R = F[t]/(t^k)$. Let $\mathcal{I}_{r,k}^{m,n}$ be the ideal of $F[\{x_{i,j}^{(l)},\ 1 \leq i \leq m,\ 1 \leq j \leq n,\ 0 \leq l < k\}]$ generated by the coefficients of powers of t in each $r \times r$ minor of the generic matrix $X(t)$, and define $\mathcal{Z}_{r,k}^{m,n} \subseteq \mathbf{A}_F^{nmk}$ to be the zero set of $\mathcal{I}_{r,k}^{m,n}$. These varieties $\mathcal{Z}_{r,k}^{m,n}$ are therefore natural generalizations of the classical varieties $\mathcal{Z}_r^{m,n}$, and when $k = 1$, of course, we simply recover the original $\mathcal{Z}_r^{m,n}$.

Our interest in these varieties arises from previous work on commuting triples of matrices. In the paper [8], the second author and Neubauer determined the variety of commuting pairs in the centralizers of 2-regular matrices (a matrix is said to be r-regular if each eigenspace is at most r dimensional). They observed there that when C is a 2-regular $n \times n$ matrix, the variety of commuting pairs in the centralizer of C is the product of $\mathbf{A}_F^p$ (for suitable p) and the subvariety of 2×3 matrices over $F[t]/(t^k)$ where the coefficients of powers of t of all 2×2 minors vanish. This second factor is of course just the variety $\mathcal{Z}_{2,k}^{2,3}$ introduced above. It was then natural to recognize $\mathcal{Z}_{2,k}^{2,3}$ as belonging to the larger class of varieties $\mathcal{Z}_{r,k}^{m,n}$, and to study this larger family.

It is clear that $\mathcal{Z}_{r,k}^{m,n}$ consists of the classical variety $\mathcal{Z}_r^{m,n}$, and at each point of $\mathcal{Z}_r^{m,n}$, those parameterized degree $k-1$ curves in $\mathbf{A}_F^{mn}$ vanishing on $\mathcal{Z}_r^{m,n}$ at that point to order k. Thus, these varieties are just the spaces of $k-1$-th order jets of the classical varieties. (In particular, when $k = 2$, $\mathcal{Z}_{r,2}^{m,n}$ may be considered as the "tangent bundle" to the classical determinantal variety $\mathcal{Z}_r^{m,n}$. Of course, this is not really a bundle in the usual sense, since different fibers will have different dimensions.) From another point of view, the varieties $\mathcal{Z}_{r,k}^{m,n}$ are the restriction (or direct image) from R to F, in the sense of Weil ([4, I, §1, 6.6]) of the scalar extension of the classical varieties $\mathcal{Z}_r^{m,n} \times_F R$.

Since in general the fibers over the base $\mathcal{Z}_r^{m,n}$ will not all be of the same dimension, it is not a priori clear whether the assemblage of the base space and its fibers should be reducible or irreducible. We show here that if we set all maximal minors to zero (i.e., $r = m$), then $\mathcal{Z}_{m,k}^{m,n}$ is indeed irreducible, but if we consider *nonmaximal* minors (i.e., $2 \le r \le m-1$, the case $r = 1$ being trivial), then $\mathcal{Z}_{r,k}^{m,n}$ breaks up into several irreducible components, not all of the same dimension. We determine these components completely for the 2×2 minors case (for any k); there are exactly $\lfloor k/2 \rfloor + 1$ components. In general, we show that when $r < m$, the minimum number of irreducible components must be $\lfloor k/2 \rfloor + 1$. For small values of k ($k = 2$ and 3) we determine all the components of $\mathcal{Z}_{r,k}^{m,n}$ (for $k = 3$ for all but finitely many (m, n) pairs).

In the special case where $m = n$ and where we consider the maximal minor, we also show that the defining equations form a Groebner basis (with respect to a suitable ordering) for the ideal generated by these equations, and we are then able to show that the ideal is actually prime. It follows easily that the coordinate ring is a complete intersection ring.

We introduce here some alternative notation for the entries of the generic matrix that will be of much use: We will denote the i-th row of the matrix $X(t)$ over $F[t]/(t^k)$ by $\mathbf{u}_i(t)$: this is an element of $(F[t]/t^k)^n$. We will write $\mathbf{u}_i(t) = \sum_{l=0}^{k-1} \mathbf{u}_i^{(l)} t^l$, so the various $\mathbf{u}_i^{(l)}$ are row vectors from F^n. We will sometimes refer to $\mathbf{u}_i^{(l)}$ by itself as the "row $\mathbf{u}_i^{(l)}$." We will also refer to a vector of the form $\mathbf{u}_i^{(l)}$ as being "of degree l." In a similar vein, we will talk of a variable of the form $x_{i,j}^{(l)}$ as being of "of degree l." In particular, when we talk of a "degree zero" minor, we will mean a minor of the matrix $X(0) = ((x_{i,j}^{(0)}))$.

We also note that the paper [9] discusses a related set of objects: the quantum Grassmannians, whose coordinate rings are the subalgebras of $F[x_{i,j}^{(l)}]$ generated by the coefficients of powers of t of the various $m \times m$ minors of an $m \times n$ matrix.

2. The Fundamental Reduction Process

We describe here a reduction process that exhibits our varieties $\mathcal{Z}_{r,k}^{m,n}$ to be a union of two subvarieties, one isomorphic to $\mathcal{Z}_{r,k-r}^{m,n} \times \mathbf{A}^{mn(r-1)}$ (or to $\mathbf{A}^{mn(k-1)}$ when $k \leq r$), and another whose components are in one-to-one correspondence with the components of $\mathcal{Z}_{r-1,k}^{m-1,n-1}$, and which, in fact, is birational to $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbf{A}^{(m+n-1)k}$. This reduction will be a key tool in understanding the components of $\mathcal{Z}_{r,k}^{m,n}$. (We assume throughout that $r \geq 2$.)

Lemma 2.1. *The subvariety of $\mathcal{Z}_{r,k}^{m,n}$ where all $x_{i,j}^{(0)}$ are zero is isomorphic to $\mathcal{Z}_{r,k-r}^{m,n} \times \mathbf{A}^{mn(r-1)}$ when $k > r$, and isomorphic to $\mathbf{A}^{mn(k-1)}$ when $k \leq r$.*

Proof. This can be seen easily by writing the equations defining $\mathcal{Z}_{r,k}^{m,n}$ in terms of the rows $\mathbf{u}_i^{(l)}$. Our determinantal equations read

$$\mathbf{u}_{i_1} \wedge \mathbf{u}_{i_2} \wedge \cdots \wedge \mathbf{u}_{i_r} = \mathbf{0} \tag{1}$$

for all $1 \leq i_1 < i_2 < \cdots < i_r \leq m$. This is an equation in $\bigwedge^r (F[t]/(t^k))^n$, which expands to a set of k equations in $\bigwedge^r F^n$, one for the coefficient of each power of t. The equation for the coefficient of t^l reads

$$\sum_{d_1+d_2+\cdots+d_r=l} \mathbf{u}_{i_1}^{(d_1)} \wedge \mathbf{u}_{i_2}^{(d_2)} \wedge \cdots \wedge \mathbf{u}_{i_r}^{(d_r)} = \mathbf{0}, \quad l = 0, \ldots, k-1. \tag{2}$$

It is clear that if all $\mathbf{u}_i^{(0)}$ are zero, then all terms in the coefficients of t^l, for $l = 0, 1, \ldots, r-1$ become zero, since any r-fold product of degree at most $r-1$ must contain at least one term of degree 0. If $k \leq r$, this just means that there are no equations governing the remaining variables $x_{i,j}^{(l)}$ for $l \geq 1$, so the subvariety is isomorphic to $\mathbf{A}^{mn(k-1)}$. When $k > r$, the equation for the coefficient of t^l for $l = r, r+1, \ldots, k-1$ now reads (after all terms involving any $\mathbf{u}_i^{(0)}$ have been removed)

$$\sum_{\substack{d_1+d_2+\cdots+d_r=l \\ d_1\geq 1,\ldots,d_r\geq 1}} \mathbf{u}_{i_1}^{(d_1)} \wedge \mathbf{u}_{i_2}^{(d_2)} \wedge \cdots \wedge \mathbf{u}_{i_r}^{(d_r)} = \mathbf{0}, \quad l = r, \ldots, k-1. \tag{3}$$

Observe that none of the rows $\mathbf{u}_i^{(k-1)}$, $\mathbf{u}_i^{(k-2)}$, $\ldots$, $\mathbf{u}_i^{(k-(r-1))}$ show up in these equations. For, every summand is an r-fold wedge product of degree l, and if, for instance, $\mathbf{u}_i^{(k-(r-1))}$ were to appear in a summand, then the minimum degree of that summand would be $k-(r-1)+(r-1) = k > k-1$. Thus, there are no equations governing the variables $x_{i,j}^{(l)}$ for $k-(r-1) \leq l \leq k-1$, which accounts for the factor $\mathbf{A}^{mn(r-1)}$. Setting $e_i = d_i - 1$, these equations can be rewritten as

$$\sum_{\substack{e_1+e_2+\cdots+e_r=l-r \\ e_i\geq 0}} \mathbf{u}_{i_1}^{(e_1+1)} \wedge \mathbf{u}_{i_2}^{(e_2+1)} \wedge \cdots \wedge \mathbf{u}_{i_r}^{(e_r+1)} = \mathbf{0}, \quad l = r, \ldots, k-1, \tag{4}$$

or what is the same thing,

$$\sum_{\substack{e_1+e_2+\cdots+e_r=l' \\ e_i\geq 0}} \mathbf{u}_{i_1}^{(e_1+1)} \wedge \mathbf{u}_{i_2}^{(e_2+1)} \wedge \cdots \wedge \mathbf{u}_{i_r}^{(e_r+1)} = \mathbf{0}, \quad l' = 0, \ldots, k-r-1. \tag{5}$$

But these are precisely the equations that one would obtain if one were to consider the generic matrix $m \times n$ matrix with rows $\mathbf{u}_i^{(1)} + \mathbf{u}_i^{(2)}t + \cdots + \mathbf{u}_i^{(k-r)}t^{k-r-1}$ $(1 \leq i \leq m)$ and set determinants of $r \times r$ minors to zero modulo t^{k-r}. This proves the lemma. □

Our next theorem will be crucial to understanding the closure of the open set where at least one $x_{i,j}^{(l)}$ is nonzero. It is merely an extension to the case $k > 1$ of the well known result in the classical case (see [2, Prop. 2.4], for instance).

We first need the following elementary remark:

Remark 2.2. Let R be a ring. For every f in $R[t]/(t^k)$ (or $R[t]$) we denote by $c_i(f)$ the coefficient of t^i in f. Let I be an ideal of $R[t]/(t^k)$ (or $R[t]$). We write $c_i(I)$ for the set of all the $c_i(f)$ with $f \in I$ and $C_k(I)$ for the union of the sets $c_i(I)$ with $0 \leq i < k$. Then $c_i(I)$ and $C_k(I)$ are ideals of R. Furthermore if $f_1, f_2, .., f_p$ generate I then the $c_i(f_j)$, $j = 1, 2, \ldots, p$, $i = 0, 1, \ldots, k-1$, generate $C_k(I)$. In particular, if $g_1, g_2, \ldots, g_q$ also generate I, then the ideal generated by the $c_i(f_j)$,

$j = 1, 2, \ldots, p$, $i = 0, 1, \ldots, k-1$, equals the ideal generated by the $c_i(g_j)$, $j = 1, 2, \ldots, q$, $i = 0, 1, \ldots, k-1$.

Write $S = F[x_{i,j}^{(l)}]$, and observe that in the ring $S[(x_{m,n}^{(0)})^{-1}][t]/(t^k)$, the element $x_{m,n}(t)$ is invertible. If we were to perform row reduction in $S[(x_{m,n}^{(0)})^{-1}][t]/(t^k)$ on the matrix X to bring all the elements in the last column above $x_{m,n}(t)$ to zero, we would subtract from row $\mathbf{u}_i$ the row $\mathbf{u}_m$ multiplied by $x_{m,n}^{-1}(t)x_{i,n}(t)$. Thus, we would replace X by a matrix

$$Y = (y_{i,j}(t)),$$

where

(6)

$$y_{i,j}(t) = \begin{cases} x_{i,j}(t) - x_{m,j}(t)x_{i,n}(t)x_{m,n}^{-1}(t) & \text{for } 1 \le i \le m-1,\ 1 \le j \le n-1 \\ 0 & \text{for } j = n \text{ and } 1 \le i \le m-1 \\ x_{i,j}(t) & \text{for } i = m \text{ and } 1 \le j \le n \end{cases}$$

Since the inverse of $x_{m,n}(t)$ can be written as a polynomial in the various entries $x_{m,n}^{(l)}$ (for $1 \le l \le k-1$) and various negative powers of $x_{m,n}^{(0)}$, we find that each $y_{i,j}^{(l)}$, for $1 \le i \le m-1$, $1 \le j \le n-1$ can be written in terms of the $x_{i,j}^{(l)}$ in the form

$$y_{i,j}^{(l)} = x_{i,j}^{(l)} - q_{i,j}^{(l)}(x_{m,j}^{(p)}, x_{i,n}^{(r)}, x_{m,n}^{(s)}, (x_{m,n}^{(0)})^{-1}), \tag{7}$$

for a suitable polynomial expression $q_{i,j}^{(l)}$ in the indicated variables ($0 \le p, r < l$, $1 \le s < l$).

With the observations above about row reduction as our motivation, and continuing with the same notation, we have the following:

Theorem 2.3. *(see* [2, Prop. 2.4]*) Assume $r \ge 2$. Let $z_{i,j}^{(l)}$, $1 \le i \le m-1$, $1 \le j \le n-1$, $0 \le l < k$ be a new set of variables, and write T for the ring $F[z_{i,j}^{(l)}]$, and T' for the ring $F[z_{i,j}^{(l)}, x_{1,n}^{(l)}, \ldots, x_{m,n}^{(l)}, x_{m,1}^{(l)}, \ldots, x_{m,n-1}^{(l)}]$ ($0 \le l < k$). Also, write Z for the $m-1 \times n-1$ matrix $(z_{i,j}(t))$ over $T[t]/(t^k)$, where $z_{i,j}(t) = \sum_{l=0}^{k-1} z_{i,j}^{(l)} t^l$. We have an isomorphism*

$$S[(x_{m,n}^{(0)})^{-1}] \cong T'[(x_{m,n}^{(0)})^{-1}],$$

given by

$$\begin{aligned} f\colon \quad x_{i,n}^{(l)} &\to x_{i,n}^{(l)} \ 1 \le i \le m \\ x_{m,j}^{(l)} &\to x_{m,j}^{(l)} \ 1 \le j \le n-1 \\ x_{i,j}^{(l)} &\to z_{i,j}^{(l)} + q_{i,j}^{(l)}(x_{m,j}^{(p)}, x_{i,n}^{(r)}, x_{m,n}^{(s)}, (x_{m,n}^{(0)})^{-1}),\ 0 \le p, r < l,\ 1 \le s < l, \\ & \quad \textit{for } 1 \le i \le m-1,\ 1 \le j \le n-1,\ 0 \le l < k. \end{aligned}$$

Under this isomorphism, the localization of $\mathcal{I}_{r,k}^{m,n}$ *at* $(x_{m,n}^{(0)})^{-1}$ *corresponds to the localization of the ideal* $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$ *at* $(x_{m,n}^{(0)})^{-1}$*, where* $\mathcal{I}_{r-1,k}^{m-1,n-1}$ *is the ideal of* T *determined by the coefficients of powers of* t *of the various* $(r-1)\times(r-1)$ *minors of the matrix* Z*. Moreover, this induces a one-to-one correspondence between the prime ideals* P *of* S *that are minimal over* $\mathcal{I}_{r,k}^{m,n}$ *and do not contain* $x_{m,n}^{(0)}$ *and the prime ideals* Q *of* T *that are minimal over* $\mathcal{I}_{r-1,k}^{m-1,n-1}$*. If* P *corresponds to* Q *in this correspondence, then the codimension of* P *in* S *equals the codimension of* Q *in* T*.*

Proof. The fact that f is an isomorphism is clear, since the map

$$\begin{array}{rrcl}\tilde{f}\colon & x_{i,n}^{(l)} & \to & x_{i,n}^{(l)}\ 1\le i\le m\\ & x_{m,j}^{(l)} & \to & x_{m,j}^{(l)}\ 1\le j\le n-1\\ & z_{i,j}^{(l)} & \to & x_{i,j}^{(l)}-q_{i,j}^{(l)}(x_{m,j}^{(p)},x_{i,n}^{(r)},x_{m,n}^{(s)},(x_{m,n}^{(0)})^{-1}),\ 0\le p,r<l,\ 1\le s<l,\\ & & & \text{for } 1\le i\le m-1,\ 1\le j\le n-1,\ 0\le l<k\end{array}$$

provides the necessary inverse.

As for the second assertion, write I for the localization of $\mathcal{I}_{r,k}^{m,n}$ at $(x_{m,n}^{(0)})^{-1}$ and J for the localization of the $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$ at $(x_{m,n}^{(0)})^{-1}$. We wish to show that $f(I)=J$. Subtracting a multiple of the m-th row from the i-th row of a matrix preserves the ideal of $S[(x_{m,n}^{(0)})^{-1}][t]/(t^k)$ generated by $r\times r$ minors, so by Remark 2.2, the coefficients of powers of t of the various $r\times r$ minors of the matrix Y can be taken as the generators of I. Write $\tilde{Y}$ for the upper-left $m-1\times n-1$ block of Y. Recall that $r\ge 2$. The $r\times r$ minors of Y fall into two classes. The first class consists of minors that involve the last column of Y, so by Laplace expansion, these minors are either zero or of the form $x_{m,n}(t)$ times an $(r-1)\times(r-1)$ minor of $\tilde{Y}$. Since $x_{m,n}(t)$ is invertible, we find that up to multiplication by a unit, this class of generators is precisely the set of all $(r-1)\times(r-1)$ minors of $\tilde{Y}$. The second class of generators of I consists of minors that do not involve the last column of Y. By Laplace expansion along the last row if necessary, these minors can be written as $S[(x_{m,n}^{(0)})^{-1}]$ linear combinations of suitable $(r-1)\times(r-1)$ minors of $\tilde{Y}$. It follows that I is generated precisely by the set of all $(r-1)\times(r-1)$ minors of $\tilde{Y}$. On the other hand, J is generated by all $(r-1)\times(r-1)$ minors of the matrix Z. Thus, under the map f, these generators of I map precisely to generators of J, so $f(I)=J$.

As for the last assertion, we have a one-to-one correspondence between the minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that do not contain $x_{m,n}^{(0)}$ and the minimal primes of the localized ideal I in $S[(x^{(0)}_{m,n})^{-1}]$. By the isomorphism described above, these are in one-to-one correspondence with the minimal primes of the ideal J of $T'[(x^{(0)}_{m,n})^{-1}]$. These, in turn, are in one-to-one correspondence with the minimal primes of the ideal $\mathcal{I}^{m-1,n-1}_{r-1,k}T'$ of T' that do not contain $x^{(0)}_{m,n}$. But since T' is just an extension of T obtained by adding the indeterminates $x^{(l)}_{1,n}, \ldots, x^{(l)}_{m,n}$, $x^{(l)}_{m,1}, \ldots, x^{(l)}_{m,n-1}$, $(0 \le l < k)$, the minimal primes of $\mathcal{I}^{m-1,n-1}_{r-1,k}T'$ are in one-to-one correspondence with the minimal primes of $\mathcal{I}^{m-1,n-1}_{r-1,k}$ in T; specifically, the minimal prime Q of $\mathcal{I}^{m-1,n-1}_{r-1,k}$ corresponds to $Q[x^{(l)}_{1,n}, \ldots, x^{(l)}_{m,n}, x^{(l)}_{m,1}, \ldots, x^{(l)}_{m,n-1}]$. Moreover, tracing through this correspondence, since localization and adding indeterminates does not change the codimension of a prime ideal that avoids the localization set, we find that the correspondence preserves the codimension of the respective prime ideals in their respective rings. This gives us the assertion. □

Remark 2.4. The theorem above shows that there is a birational isomorphism between the closure in $\mathcal{Z}^{m,n}_{r,k}$ of where $x^{(0)}_{m,n} \neq 0$ and $\mathcal{Z}^{m-1,n-1}_{r-1,k} \times \mathbf{A}^{k(m+n-1)}$, with the domains of definition being the open set of $\mathcal{Z}^{m,n}_{r,k}$ where $x^{(0)}_{m,n} \neq 0$ and the open set of $\mathcal{Z}^{m-1,n-1}_{r-1,k} \times \mathbf{A}^{k(m+n-1)}$ where the "free" variable $x^{(0)}_{m,n} \neq 0$.

Remark 2.5. Notice that there is nothing special in these considerations about the variable $x^{(0)}_{m,n}$. Essentially the same result holds for localization at any other variable $x^{(0)}_{i,j}$. The matrix $\tilde{Y}$ in that case would arise from the removal of the j-th column and i-th row of the matrix X.

The ideas in the proof of the theorem above also lead to the following:

Proposition 2.6. *Let $S = F[x^{(l)}_{i,j}]$ as before. Let P be a minimal prime ideal of $\mathcal{I}^{m,n}_{r,k}$. Then for any two pairs of indices (i,j) and (i',j'), P contains $x^{(0)}_{i,j}$ iff it contains $x^{(0)}_{i',j'}$.*

Proof. It is sufficient to prove this for the case where $(i',j') = (m,n)$. Let P be a prime ideal minimal over $\mathcal{I}^{m,n}_{r,k}$ that does not contain $x^{(0)}_{m,n}$. Assume to the contrary that it contains $x^{(0)}_{i,j}$. Then the localization $\tilde{P}$ of P at $x^{(0)}_{m,n}$ will also contain $x^{(0)}_{i,j}$, and will be minimal over the localization of I of $\mathcal{I}^{m,n}_{r,k}$. Hence, the ideal $f(\tilde{P})$, where f is as in Theorem 2.3, will contain $f(x^{(0)}_{i,j}) = z^{(0)}_{i,j} + q^{(0)}_{i,j}$ $(= x^{(0)}_{i,j}$ in the case $i = m$ or $j = n)$, and will be minimal over $J = f(I)$. But as is readily seen, $q^{(0)}_{i,j}$ is just

$x_{i,n}^{(0)}x_{m,j}^{(0)}(x_{m,n}^{(0)})^{-1}$. Since $x_{m,n}^{(0)}$ is a unit, it follows that $f(P)$ will contain $x_{m,n}^{(0)}z_{i,j}^{(0)}+x_{i,n}^{(0)}x_{m,j}^{(0)}$ (in the case $i \neq m$ and $j \neq n$). Under the localization map from T' to $T'[(x_{m,n}^{(0)})^{-1}]$, $f(P)$ will correspond to a prime ideal Q' of T', that is minimal over the ideal $\mathcal{I}_{r-1,k}^{m-1,n-1}T'$. Moreover, Q' will contain $x_{m,n}^{(0)}z_{i,j}^{(0)}+x_{i,n}^{(0)}x_{m,j}^{(0)}$ ($x_{i,j}^{(0)}$ in the case $i = m$ or $j = n$). As we saw in the proof of the last assertion of Theorem 2.3 above, Q' must be of the form $Q[x_{1,n}^{(l)}, \dots, x_{m,n}^{(l)}, x_{m,1}^{(l)}, \dots, x_{m,n-1}^{(l)}]$ for some minimal prime Q of the ideal $\mathcal{I}_{r-1,k}^{m-1,n-1}$ of T. But this is impossible, since the coefficient of $x_{i,n}^{(0)}x_{m,j}^{(0)}$ in the element $x_{m,n}^{(0)}z_{i,j}^{(0)} + x_{i,n}^{(0)}x_{m,j}^{(0)}$, namely, 1, is not in Q (and similarly, the coefficient of $x_{i,j}^{(0)}$ is not in Q in the case $i = m$ or $j = n$). Hence P cannot contain $x_{i,j}^{(0)}$.

Conversely, if P is a minimal prime ideal of $\mathcal{I}_{r,k}^{m,n}$ that does not contain $x_{i,j}^{(0)}$ but contains $x_{m,n}^{(0)}$, then the same argument as above, applied to the corresponding isomorphism obtained on localizing at $x_{i,j}^{(0)}$ (see Remark (2.5) above), gives us a contradiction. This proves the corollary. □

We are now ready to decompose our variety $\mathcal{Z}_{r,k}^{m,n}$ as described at the beginning of this section. Let Z_0 represent the union of the zero sets of all those minimal prime ideals of $\mathcal{I}_{r,k}^{m,n}$ in $S = F[x_{i,j}^{(l)}]$ that do not contain some $x_{i,j}^{(0)}$ (and hence, by Proposition 2.6 above, do not contain any $x_{i,j}^{(0)}$ for $1 \leq i \leq m$, $1 \leq j \leq n$). Z_0 is not empty: as there are clearly points in our variety where $x_{i,j}^{(0)}$ is not zero. The following is elementary:

Lemma 2.7. *For any pair (i, j), let $U_{i,j}$ represent the open set of $\mathcal{Z}_{r,k}^{m,n}$ where $x_{i,j}^{(0)} \neq 0$, and let U represent the open set of $\mathcal{Z}_{r,k}^{m,n}$ where all $x_{i,j}^{(0)}$ (for $1 \leq i \leq m$, $1 \leq j \leq n$) are nonzero. Then $Z_0 = \overline{U_{i,j}} = \overline{U}$, where the bar represents the closure of the respective sets.*

Proof. It is clear that $U \subset U_{i,j} \subset Z_0$, from which it follows that $\overline{U} \subset \overline{U_{i,j}} \subset Z_0$. For any minimal prime P of $\mathcal{I}_{r,k}^{m,n}$ that does not contain $x_{i,j}^{(0)}$, let $Z(P)$ denote its zero set. Then $U_{i,j} \cap Z(P)$ is nonempty, since otherwise, $x_{i,j}^{(0)} \in P$, and $U \cap Z(P)$ is nonempty, since otherwise, $\prod_{i,j} x_{i,j}^{(0)} \in P$. Hence, $U_{i,j} \cap Z(P)$ and $U \cap Z(P)$ are dense in $Z(P)$, so $\overline{U_{i,j}}$ and $\overline{U}$ must contain all of $Z(P)$. □

Now let Z_1 represent the subvariety of $\mathcal{Z}_{r,k}^{m,n}$ where all $x_{i,j}^{(0)}$, for $1 \leq i \leq m$, $1 \leq j \leq n$, are zero. Note that Z_1 may be contained wholly in

Z_0; this will happen if the prime ideals of S that contain $\mathcal{I}_{r,k}^{m,n} + x_{i,j}^{(0)}S$ are not minimal over $\mathcal{I}_{r,k}^{m,n}$, i.e., no minimal prime ideal of $\mathcal{I}_{r,k}^{m,n}$ contains any $x_{i,j}^{(0)}$.

The following is just a summary of our discussions in this section:

Theorem 2.8. *The variety $\mathcal{Z}_{r,k}^{m,n}$ (for $r \geq 2$) is the union of two subvarieties Z_0 and Z_1. The variety Z_0 is the closure of any of the open sets $U_{i,j}$ ($1 \leq i \leq m$, $1 \leq j \leq n$) where $x_{i,j}^{(0)}$ is nonzero (as also the closure of the open set U where all $x_{i,j}^{(0)}$ are nonzero). Z_0 is also the union of the components of $\mathcal{Z}_{r,k}^{m,n}$ that correspond to minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that do not contain some (hence any) $x_{i,j}^{(0)}$. Such components always exist, and are in one-to-one correspondence with the components of the variety $\mathcal{Z}_{r-1,k}^{m-1,n-1}$. The correspondence preserves the codimension (in $\mathbf{A}^{mnk}$ and $\mathbf{A}^{(m-1)(n-1)k}$ respectively) of the components, and in fact, Z_0 is birational to $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbf{A}^{(m+n-1)k}$. The variety Z_1 is the subvariety of $\mathcal{Z}_{r,k}^{m,n}$ where all $x_{i,j}^{(0)}$ are zero, and is isomorphic to $\mathcal{Z}_{r,k-r}^{m,n} \times \mathbf{A}^{mn(r-1)}$ when $k > r$, and isomorphic to $\mathbf{A}^{mn(k-1)}$ when $k \leq r$. Z_1 will be wholly contained in Z_0 precisely when there are no minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that contain some (hence all) $x_{i,j}^{(0)}$.*

Remark 2.9. If there exist minimal primes of $\mathcal{I}_{r,k}^{m,n}$ that contain some (hence all) $x_{i,j}^{(0)}$, then these will correspond to some (possibly even all) components of Z_1.

3. The Case of Maximal Minors

When $r = m$, i.e., when we consider the situation where we set all maximal minors to zero, we have the following easy result:

Theorem 3.1. *The varieties $\mathcal{Z}_{m,k}^{m,n}$ are all irreducible, of codimension $k(n-m+1)$.*

Proof. We prove the irreducibility by induction on m. If $m = 1$, then the varieties $\mathcal{Z}_{1,k}^{1,n}$ (for any n and k) are clearly irreducible, in fact, $\mathcal{Z}_{1,k}^{1,n}$ is just the origin in $\mathbf{A}^{nk}$. So assume that $\mathcal{Z}_{m-1,k}^{m-1,n-1}$ is irreducible. Then there is only one minimal prime ideal lying over the ideal $\mathcal{I}_{r-1,k}^{m-1,n-1}$ in the ring $T = F[z_{i,j}^{(l)}]$ (see the statement of Theorem 2.3 for notation). Tracing through the isomorphism of Theorem 2.3 above, the localization of $\mathcal{I}_{r,k}^{m,n}$ at $x_{m,n}^{(0)}$ has only one minimal prime ideal, so in particular, there is only one minimal prime ideal of $\mathcal{I}_{r,k}^{m,n}$ in $S = F[x_{i,j}^{(l)}]$ that does not contain $x_{m,n}^{(0)}$. As in the discussion in Section 2 (in particular, see

Theorem 2.8), this means that subvariety Z_0 is irreducible. It is now sufficient to show that $Z_1 \subset Z_0$. We will do this by showing that each point in Z_1 is on a line, all but a finite number of points of which lie inside one of the open sets $U_{i,j}$. Since Z_0 is the closure of any of the open sets $U_{i,j}$, this will establish that $Z_1 \subset Z_0$.

Let Q be a point in Z_1. If Q is the origin in $\mathbf{A}^{mnk}$, then Q lies on the line λP ($\lambda \in F$) for any P in any $U_{i,j}$. (Recall that $U_{i,j}$ is nonempty.) Since for $\lambda \neq 0$ the point λP is in $U_{i,j}$, our point Q must lie in the closure of $U_{i,j}$.

Now assume Q is not the origin. In the representation of Q as rows $(\mathbf{u}_1(t), \ldots, \mathbf{u}_m(t))^T$, with $\mathbf{u}_i(t) = \sum_l \mathbf{u}_i^{(l)} t^l$ (see the notation introduced just before Lemma 2.1), $\mathbf{u}_i^{(0)} = 0$ for $i = 1, \ldots, m$. Since Q is not the origin, some $\mathbf{u}_i^{(s)} \neq 0$ for some i with $1 \leq i \leq m$, and some s with $1 \leq s < k$ and with s minimal for this i. Write $\mathbf{v}(t)$ for the vector $\mathbf{u}_i^{(s)} + \mathbf{u}_i^{(s+1)} t + \cdots + \mathbf{u}_i^{(k-1)} t^{k-s-1}$ in $(F[t]/(t^k))^n$. Consider the point $P(\lambda) = (\mathbf{u}_1(t), \ldots, \mathbf{u}_i(t), \mathbf{u}_{i+1}(t) + \lambda \mathbf{v}(t), \mathbf{u}_{i+2}(t) \ldots, \mathbf{u}_m(t))^T$ (with the understanding that if $i = m$, then $P(\lambda) = (\mathbf{u}_1(t), \ldots, \mathbf{u}_{m-2}(t), \mathbf{u}_{m-1}(t) + \lambda \mathbf{v}(t), \mathbf{u}_m(t))^T$. Then the m-fold wedge product of these vectors contains two summands: $\mathbf{u}_1(t) \wedge \cdots \wedge \mathbf{u}_i(t) \wedge \mathbf{u}_{i+1}(t) \wedge \cdots \wedge \mathbf{u}_m(t)$, and $\lambda \mathbf{u}_1(t) \wedge \cdots \wedge \mathbf{u}_i(t) \wedge \mathbf{v}(t) \wedge \cdots \wedge \mathbf{u}_m(t)$ (suitably modified if $i = m$). The first is zero, since Q is in $\mathcal{Z}_{m,k}^{m,n}$, and the second is zero since $\mathbf{u}_i(t) = t^s \mathbf{v}(t)$. Thus, the point $P(\lambda)$ is in $\mathcal{Z}_{m,k}^{m,n}$. When $\lambda \neq 0$, $P(\lambda)$ is actually in $U_{i,l}$ for some l (corresponding to any one coordinate of $\mathbf{u}_i^{(s)}$ that is nonzero). Hence, the point $Q = P(0)$ is in the closure of $U_{i,l}$, which is Z_0.

As for the codimension, the codimension of $\mathcal{I}_{m,k}^{m,n}$ is the codimension of its unique minimal prime ideal. Tracing through the localization correspondence of Theorem 2.3, this is the same as the codimension of $\mathcal{I}_{m-1,k}^{m-1,n-1}$. Continuing this localizing process, we find that the codimension of $\mathcal{I}_{m,k}^{m,n}$ is the same as the codimension of $\mathcal{I}_{1,k}^{1,n-m+1}$, which is clearly $k(n - m + 1)$. □

Remark 3.2. This proof technique breaks down when considering r-fold wedge products with $r < m$: if one were to consider a wedge product that includes the i-th row but not the $(i+1)$-th row (or contains the m-th row but not the $(m-1)$-th row if $i = m$), then the second summand of the wedge product need not be zero, so the point $P(\lambda)$ need not be in our variety at all.

3.1. **Square Matrices.** When $m = n$, i.e., when our matrices are square, and when we are still in the situation of maximal minors, we can say considerably more. Let us denote the coefficients of powers of t of the determinant of our square matrix $X(t)$ by d_l, $l = 0, \ldots, k-1$. It is easy to determine the structure of the polynomial expressions d_l. The first term d_0 is just the determinant of the matrix $X(0) = ((x_{i,j}^{(0)}))_{1\leq i,j\leq m}$. The remaining terms can be obtained by the following process: Every monomial appearing in d_0 is the form $m_\sigma = x_{\sigma(1),1}^{(0)} x_{\sigma(2),2}^{(0)} \ldots x_{\sigma(m),m}^{(0)}$ for some permutation σ of $\{1, 2, \ldots, m\}$. Given such a monomial, we define

$$\mu_l(m_\sigma) = \sum_{k_i\geq 0, \sum k_i = l} x_{\sigma(1),1}^{(k_1)} x_{\sigma(2),2}^{(k_2)} \ldots x_{\sigma(m),m}^{(k_m)}$$

We then find $d_l = \sum_{\sigma\in S_n} \mu_l(m_\sigma) sgn(\sigma)$.

We consider the graded reverse lexicographic ordering (*grevlex*) on the monomials on $S = F[x_{i,j}^{(l)}]$ given by the following scheme: $x_{1,1}^{(k-1)} > x_{1,2}^{(k-1)} > \cdots > x_{1,m}^{(k-1)} > x_{2,1}^{(k-1)} > \cdots > x_{m,m}^{(k-1)} > x_{1,1}^{(k-2)} > x_{1,2}^{(k-2)} > \cdots > x_{1,m}^{(k-2)} > x_{2,1}^{(k-2)} > \cdots > x_{m,m}^{(k-2)} > \cdots > x_{m,m}^{(1)} > x_{1,1}^{(0)} > \cdots > x_{1,m}^{(0)} > x_{2,1}^{(0)} > \cdots > x_{m,m}^{(0)}$.

(Recall that in the graded reverse lexicographic ordering the monomials of S are first ordered by the total degree, and for two monomials α and β of the same degree, α is greater than β if the rightmost nonzero element in $\alpha - \beta$ (with α and β thought of as elements of $\mathbb{Z}^{km^2}$) is negative—see [3, Chapter 2, §2], for instance.)

Theorem 3.3. *Under the grevlex ordering on S described above, the generators d_l, $l = 0, 1, \ldots, k-1$ of the ideal $\mathcal{I}_{m,k}^{m,m}$ form a Groebner basis for $\mathcal{I}_{m,k}^{m,m}$.*

Proof. The grevlex order is designed so as to favor monomials in which the lower order variables do not appear. It is easy to see then that the leading monomials (LM) of the various d_l are as follows:

$$\begin{aligned}
\text{LM}(d_0) &= x_{1,m}^{(0)} x_{2,m-1}^{(0)} \ldots x_{m,1}^{(0)} \\
\text{LM}(d_1) &= x_{1,m-1}^{(0)} x_{2,m-2}^{(0)} \ldots x_{m-1,1}^{(0)} x_{m,m}^{(1)} \\
\text{LM}(d_2) &= x_{1,m-2}^{(0)} x_{2,m-3}^{(0)} \ldots x_{m-2,1}^{(0)} x_{m-1,m}^{(1)} x_{m,m-1}^{(1)} \\
\vdots &= \vdots \\
\text{LM}(d_{m-1}) &= x_{1,1}^{(0)} x_{2,m}^{(1)} x_{3,m-1}^{(1)} \ldots x_{m,2}^{(1)}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{LM}(d_m) &= x^{(1)}_{1,m}x^{(1)}_{2,m-1}\cdots x^{(1)}_{m,1}\\
\mathrm{LM}(d_{m+1}) &= x^{(1)}_{1,m-1}x^{(1)}_{2,m-2}\cdots x^{(1)}_{m-1,1}x^{(2)}_{m,m}\\
\vdots &= \vdots
\end{aligned}
$$

so in general, we have

(8)

$$\mathrm{LM}(d_{\lambda m+\mu}) = x^{(\lambda)}_{1,m-\mu}x^{(\lambda)}_{2,m-\mu-1}\cdots x^{(\lambda)}_{m-\mu,1}x^{(\lambda+1)}_{m-\mu+1,m}x^{(\lambda+1)}_{m-\mu+2,m-1}\cdots x^{(\lambda+1)}_{m,m-\mu+1},$$

where $0 \le \mu \le m-1$.

It is clear that the leading monomials of d_i and d_j, for distinct i and j, are formed from sets of variables that are disjoint from one another. Hence, the leading terms of the various d_l are all pairwise relatively prime. It follows, e.g. from [3, Chapter 2, §9, Prop. 4 and Theorem 3], that the d_l form a Groebner basis for $\mathcal{I}^{m,m}_{m,k}$. □

We now have:

Theorem 3.4. *The ideals $\mathcal{I}^{m,m}_{m,k}$ are prime ideals, of codimension k. The coordinate rings of the varieties $\mathcal{Z}^{m,m}_{m,k}$ are consequently complete intersection rings, and hence Cohen-Macaulay, and the degree of $\mathcal{Z}^{m,m}_{m,k}$ is m^k.*

Proof. Since the polynomials d_l form a Groebner basis for $\mathcal{I}^{m,m}_{m,k}$, and since the lead terms of these d_l in (8) are obviously square free, it follows that the ideals $\mathcal{I}^{m,m}_{m,k}$ are radical. (This is well known and easy: if $f^r \in \mathcal{I}^{m,m}_{m,k}$, then $(\mathrm{LM}(f))^r \in <\mathrm{LM}(d_0),\dots,\mathrm{LM}(d_{k-1})>$, so $\mathrm{LM}(d_i)$ divides $(\mathrm{LM}(f))^r$ for some i. Since $\mathrm{LM}(d_i)$ is square free, this means that $\mathrm{LM}(d_i)$ divides $\mathrm{LM}(f)$, so $\mathrm{LM}(d_i)e = \mathrm{LM}(f)$ for some monomial e. Then $f - ed_i$ is also in the radical of $\mathcal{I}^{m,m}_{m,k}$, and has lower lead monomial. We proceed thus to find that f is in $\mathcal{I}^{m,m}_{m,k}$.) But we have already seen in Theorem 3.1 above that the radical of $\mathcal{I}^{m,m}_{m,k}$ must be prime. It follows that the ideals $\mathcal{I}^{m,m}_{m,k}$ are prime and that their codimension is equal to k. Then, by the definition of a complete intersection ring, it follows that the coordinate ring $S/\mathcal{I}^{m,m}_{m,k}$ of the variety $\mathcal{Z}^{m,m}_{m,k}$ is a complete intersection ring. By e.g. [5, Chapter 18, Prop. 18.13], it is Cohen-Macaulay. As for the degree, this is standard (see for instance [6, Chap. III, §3.5], where the Hilbert polynomial of a complete intersection is also computed). □

Remark 3.5. Theorem 3.4 would suggest that in general the varieties $\mathcal{Z}^{m,n}_{m,k}$ corresponding to the maximal minors are Cohen-Macaulay and that their ideals $\mathcal{I}^{m,n}_{m,k}$ are prime and have Groebner bases with square-free leading terms. We wish to explore some of these in future work; the special case of $m = 2$ will appear in [7].

4. Equations for Some Open Sets of $\mathcal{Z}^{m,n}_{r,k}$

In this section, we will derive the key equations that will hold in certain open sets of our variety and will enable us to show that our varieties are reducible when $r < m$ (i.e., we set nonmaximal minors to zero).

We start with an elementary and well-known result:

Lemma 4.1. *Let R be a commutative ring, R^* its group of invertible elements, and R^n the free module of rank n. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1} \in R^n$ are such that for some $\mathbf{w}_1, \ldots, \mathbf{w}_{n-r+1} \in R^n$, the product $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{n-r+1} \in R^*$ (i.e., the elements $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ can be extended to a basis of R^n). If $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{v} = 0$ for some element $\mathbf{v} \in R^n$, then $\mathbf{v} = \sum_{i=1}^{r-1} \alpha_i \mathbf{u}_i$ for some $\alpha_i \in R$.*

Proof. Since $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{n-r+1}$ is the determinant of the matrix $[\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-r+1}]$, the hypothesis shows that this matrix is invertible in $End(R^n)$. Hence we can find the unique solution $\mathbf{x} = (\alpha_1, \ldots, \alpha_n)^T$ to the equation

$$[\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-r+1}]\mathbf{x} = \mathbf{v}$$

by Cramer's rule. But the assumption $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{v} = 0$ shows that α_j must be zero for $j = r, \ldots, n$, since for such j, one of the $\mathbf{w}_i$ will be replaced by v during the solution. Hence $\mathbf{v} = \sum_{i=1}^{r-1} \alpha_i \mathbf{u}_i$. □

For the rest of this section we will write R for the polynomial ring $F[t]$, and write $\overline{R}$ for the ring $F[t]/(t^k)$. Let M be the free R module of rank n, and $\overline{M} = M \otimes_R F[t]/(t^k)$ be the free $\overline{R}$ module of rank n. Note that any $\mathbf{u}$ in $\overline{M}$ can be uniquely written as $\sum_{i=0}^{k-1} \mathbf{u}^{(l)} t^l \mod t^k M$ for suitable $\mathbf{u}^{(l)} \in F^n$.

Lemma 4.1 leads to the following:

Corollary 4.2. *Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \overline{M}$ are such that $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r = 0$ and $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \neq 0$ in $\bigwedge^{(r-1)} F^n$ (here, the $\mathbf{u}_j^{(0)}$ are as described in the previous paragraph). Then, there are $\alpha_1, \ldots, \alpha_{r-1} \in \overline{R}$ such that $\mathbf{u}_r = \sum_{i=1}^{r-1} \alpha_i \mathbf{u}_i$.*

Proof. Since $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \neq 0$ in $\bigwedge^{(r-1)} F^n$, there are elements $\mathbf{w}_1, \ldots, \mathbf{w}_{n-r+1} \in F^n$ such that $\mathbf{u}_1^{(0)}, \ldots, \mathbf{u}_{r-1}^{(0)}, \mathbf{w}_1, \ldots, \mathbf{w}_{n-r+1}$ form a basis for the vector space F^n. In particular, $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \wedge \mathbf{w}_1 \wedge$

$\cdots \wedge \mathbf{w}_{n-r+1} \neq 0$ in F. Thus, the constant term in the wedge product $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{n-r+1}$ will be nonzero, so $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1} \wedge \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{n-r+1}$ is in $\overline{R}^*$. The result now follows from Lemma 4.1. □

We now come to the main result that generates equations for certain open sets of our variety. For any $v \in M$, we will write $\overline{v}$ for the image of v under the map $M \mapsto \overline{M} = M/t^k M$.

Theorem 4.3. *Suppose* $\mathbf{u}_1, \ldots, \mathbf{u}_m \in R^n$ *(with* $m \leq n$*) are of degree at most* $k-1$ *(that is,* $\mathbf{u}_j = \sum_{l=0}^{k-1} \mathbf{u}_j^{(l)} t^l$*), and suppose that* $\overline{\mathbf{u}_{j_1}} \wedge \cdots \wedge \overline{\mathbf{u}_{j_r}} = 0$ *for all sequences* $1 \leq j_1 < \cdots < j_r \leq m$*. Further, assume that* $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \neq 0$*. Then*

$$\mathbf{u}_{j_1} \wedge \cdots \wedge \mathbf{u}_{j_r} \wedge \mathbf{u}_{j_{r+1}} \in t^{2k} \bigwedge^{r+1} M$$

for all sequences $1 \leq j_1 < \cdots < j_r < j_{r+1} \leq m$*.*

Proof. Applying Corollary 4.2 to the elements $\overline{\mathbf{u}_1}, \ldots, \overline{\mathbf{u}_{r-1}}, \overline{\mathbf{u}_j}$ ($r \leq j \leq m$), we find

$$\overline{\mathbf{u}_j} = \sum_{i=1}^{r-1} \alpha_{j,i} \overline{\mathbf{u}_i}$$

for elements $\alpha_{j,i}$ in $\overline{R}$. If $\alpha_{j,i} = p_{j,i}(t) + t^k R$ for uniquely determined polynomials $p_{j,i} \in R$ of degree at most $k-1$, define

$$\mathbf{v}_j = \begin{cases} \mathbf{u}_j & j = 1, \ldots, r-1, \\ \sum_{i=1}^{r-1} p_{j,i} \mathbf{u}_i & j = r, r+1, \ldots, m. \end{cases}$$

Then, since the $\mathbf{v}_j$ depend linearly on the $r-1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$, we find, this time in M, that

$$\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} \wedge \mathbf{v}_{j_{r+1}} = 0 \tag{9}$$

for all sequences $1 \leq j_1 < \cdots < j_r < j_{r+1} \leq m$.

Now, for any j, $\mathbf{u}_j$ and $\mathbf{v}_j$ are equal modulo t^k, so we may write $\mathbf{u}_j = \mathbf{v}_j + t^k \mathbf{y}_j$ for suitable $\mathbf{y}_j$. Then,

$$\mathbf{u}_{j_1} \wedge \cdots \wedge \mathbf{u}_{j_r} \wedge \mathbf{u}_{j_{r+1}} = (\mathbf{v}_{j_1} + t^k \mathbf{y}_{j_1}) \wedge \cdots \wedge (\mathbf{v}_{j_{r+1}} + t^k \mathbf{y}_{j_{r+1}}).$$

Expanding the right side, we get the following: a term $\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} \wedge \mathbf{v}_{j_{r+1}}$ which is zero by Equality (9), a sum of terms of the form $t^k(\pm \mathbf{y}_{j_i}) \wedge \mathbf{v}_{j_1} \wedge \cdots \wedge \widehat{\mathbf{v}_{j_i}} \wedge \cdots \wedge \mathbf{v}_{j_{r+1}}$ (where the hat denotes the omission of the term under the hat), and then terms that are clearly in $t^{2k} \bigwedge^{r+1} M$ or higher. But the product $\mathbf{v}_{j_1} \wedge \cdots \wedge \widehat{\mathbf{v}_{j_i}} \wedge \cdots \wedge \mathbf{v}_{j_{r+1}}$ is already in $t^k \bigwedge^r M$, since on reduction modulo t^k, we get the r-fold wedge product of the various $\overline{\mathbf{u}_j}$ which is 0 in $\bigwedge^r \overline{M}$. It follows that $\mathbf{u}_{j_1} \wedge \cdots \wedge \mathbf{u}_{j_r} \wedge \mathbf{u}_{j_{r+1}}$ is in $t^{2k} \bigwedge^{r+1} M$. This proves the theorem. □

Corollary 4.4. *Assume the $\mathbf{u}_j$ $(1 \leq j \leq m)$ are as in the preceding theorem. Then,*

$$\sum_{\substack{l_1+\cdots l_{r+1}=w \\ 0 \leq l_j < k}} \mathbf{u}_{j_1}^{(l_1)} \wedge \cdots \wedge \mathbf{u}_{j_r}^{(l_r)} \wedge \mathbf{u}_{j_{r+1}}^{(l_{r+1})} = 0 \tag{10}$$

for each w such that $0 \leq w < 2k$, and for all sequences $1 \leq j_1 < \cdots < j_r < j_{r+1} \leq m$. In particular, when $r = 2$,

$$\sum_{\substack{l_1+l_2+l_3=w \\ 0 \leq l_1, l_2, l_3 < k}} \mathbf{u}_{j_1}^{(l_1)} \wedge \mathbf{u}_{j_2}^{(l_2)} \wedge \mathbf{u}_{j_3}^{(l_3)} = 0 \tag{11}$$

on the subvariety Z_0 of $\mathcal{Z}_{2,k}^{m,n}$, for each w such that $0 \leq w < 2k$, and for all sequences $1 \leq j_1 < j_2 < j_3 \leq m$.

Proof. The expressions on the left hand side of the equalities (10) are merely the coefficients of t^w, $0 \leq w < 2k$, of $\mathbf{u}_{j_1} \wedge \cdots \wedge \mathbf{u}_{j_r} \wedge \mathbf{u}_{j_{r+1}}$, so by the previous theorem these are zero whenever $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \neq 0$. When $r = 2$, the subvariety Z_0 is the closure of the open set where $\mathbf{u}_1^{(0)} \neq 0$. □

Remark 4.5. There is another set of equations that hold on the closure of the open set of $\mathcal{Z}_{r,k}^{m,n}$ where $\mathbf{u}_1^{(0)} \wedge \cdots \wedge \mathbf{u}_{r-1}^{(0)} \neq 0$. By Corollary 4.2, all other vectors $\mathbf{u}_i$ can be expressed as an R-linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$. In particular, all other vectors $u_i^{(l)}$ can be expressed as an F-linear combination of the $k(r-1)$ vectors $\mathbf{u}_1^{(l)}, \ldots, \mathbf{u}_{r-1}^{(l)}$, $l = 0, \ldots, k-1$. It follows that the $(k(r-1)+1)$-fold wedge product of any of the vectors $\mathbf{u}_i^{(l)}$ must be zero on this closure, for $i = 1, \ldots, m$ and for $l = 0, \ldots, k-1$. This will be trivially true if $(k(r-1)+1) > n$.

5. The Case of 2×2 Nonmaximal Minors

In this section, we will completely describe the components of the variety $\mathcal{Z}_{2,k}^{m,n}$ when $n \geq m \geq 3$ and $k \geq 2$.

For this section, we will write Y_0 for our variety $\mathcal{Z}_{2,k}^{m,n}$, and X_0 for the closure Z_0 of any of the open sets $U_{i,j}$ described in Theorem 2.8. We will write Y_1 for the subvariety Z_1 of Theorem 2.8 where all $x_{i,j}^{(0)}$ are zero.

Now Y_1 is isomorphic to $\mathcal{Z}_{2,k-2}^{m,n} \times \mathbf{A}^{mn}$ when $k > 2$, and isomorphic to $\mathbf{A}^{mn}$ when $k = 2$ (Lemma 2.1). In the case $k > 2$, we will write W_1 for the factor $\mathcal{Z}_{2,k-2}^{m,n}$. (Recall from the proof of Lemma 2.1 that in the case

$k > 2$, the variety Y_1 is really determined by considering the generic $m \times n$ matrix with rows $\mathbf{u}_i^{(1)} + \mathbf{u}_i^{(2)} t + \cdots + \mathbf{u}_i^{(k-2)} t^{k-3}$ $(1 \leq i \leq m)$ and setting determinants of 2×2 minors to zero modulo t^{k-2}.)

Notice that if $k = 3$, the factor W_1 is the classical determinantal variety of 2×2 minors of the generic matrix $((x_{i,j}^{(1)}))$, and this variety is known to be irreducible ([2]). Hence, when $k = 3$, Y_1 is just a product of an irreducible variety with an affine piece, and is hence irreducible.

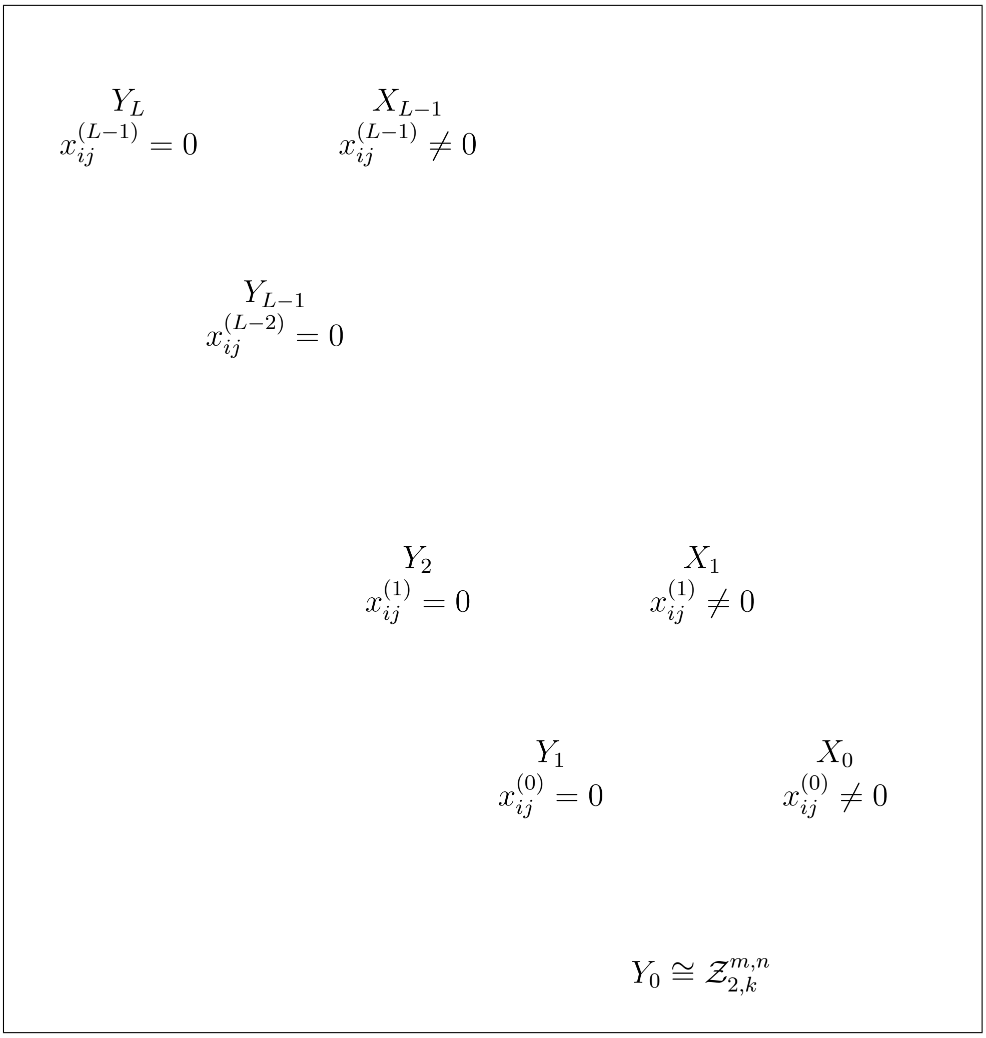

Figure 1: Irreducible components of $\mathcal{Z}_{2,k}^{m,n}$

If $k > 3$, we will write T_1 for the subvariety "Z_0" of W_1, i.e., the closure in W_1 of the open set where some $x_{i,j}^{(1)} \neq 0$. Also, we will write X_1 for $T_1 \times \mathbf{A}^{mn}$.

Write $k = 2L + 1$ or $k = 2L$ according to whether k is odd or even. Proceeding thus, we have the following subvarieties Y_s, X_s, for $s = 0, 1, \ldots, L = \lfloor k/2 \rfloor$:

- Y_0, X_0 are as already described.
- (For $0 < s < L$) Y_s is the subvariety where all row vectors $\mathbf{u}_i^{(0)}, \ldots, \mathbf{u}_i^{(s-1)}$ are zero, $1 \le i \le m$. The various rows $\mathbf{u}_i^{(s)}, \ldots, \mathbf{u}_i^{((k-1)-s)}$, $1 \le i \le m$, are governed by the condition that all 2×2 minors of the matrix with rows $\mathbf{u}_i^{(s)} + \mathbf{u}_i^{(s+1)} t + \cdots + \mathbf{u}_i^{((k-1)-s)} t^{k-2s-1}$ are zero modulo t^{k-2s}. The rows $\mathbf{u}_i^{(k-s)}, \ldots, \mathbf{u}_i^{(k-1)}$, $1 \le i \le m$, are all free. We write $Y_s \cong W_s \times \mathbf{A}^{mns}$, where $W_s \cong \mathcal{Z}^{m,n}_{2,k-2s}$. We write T_s for the subvariety "Z_0" of W_s, that is, the closure in W_s of the open set where some $x_{i,j}^{(s)} \ne 0$, and we write X_s for $T_s \cong \mathbf{A}^{mns}$.
- If k = 2L, then Y_L is given by setting all row vectors $\mathbf{u}_i^{(0)}, \ldots, \mathbf{u}_i^{(L-1)}$ to zero, $1 \le i \le m$, with all remaining row vectors $\mathbf{u}_i^{(L)}, \ldots, \mathbf{u}_i^{(k-1)}$, $1 \le i \le m$, being free. Thus, $Y_L \cong \mathbf{A}^{mnL}$.
- If k = 2L +1, then on Y_L, all row vectors $\mathbf{u}_i^{(0)}, \ldots, \mathbf{u}_i^{(L-1)}$ are zero, $1 \le i \le m$. The row vectors $\mathbf{u}_i^{(L)}$, $1 \le i \le m$, satisfy the classical determinantal equations for the 2×2 minors of the generic $m \times n$ matrix $((x_{i,j}^{(L)}))$. W_L will denote the classical determinantal variety determined by the $\mathbf{u}_i^{(L)}$. The various row vectors $\mathbf{u}_i^{(L+1)}, \ldots, \mathbf{u}_i^{(k-1)}$, $1 \le i \le m$, are all free. Thus, $Y_L \cong W_L \times \mathbf{A}^{mnL}$, and since W_L is a classical determinantal variety, Y_L is itself irreducible ([2]).

Our result is the following:

Theorem 5.1. *The variety $\mathcal{Z}^{m,n}_{2,k}$ ($n \ge m \ge 3$, $k \ge 2$) is reducible. Its irreducible components are the subvarieties X_0, X_1, ..., X_{L-1}, and Y_L described above. The components X_s, $s = 0, 1, \ldots, L-1$, have codimension $(m-1)(n-1)(k-2s) + mns$. If $k = 2L$, Y_L has codimension mnL, while if $k = 2L+1$, Y_L has codimension $(m-1)(n-1) + mnL$.*

Proof. The fact that the various X_s and Y_L are irreducible and have the stated codimension follows easily from the descriptions of the various X_s and Y_L above and from Theorem 2.3. We have seen that for $0 \le s < L$, X_s ($\cong T_s \times \mathbf{A}^{mns}$) sits in the portion of $\mathbf{A}^{mnk}$ determined by setting all rows $\mathbf{u}_i^{(l)}$ to zero, $l = 0, 1, \ldots, s-1$ (when $s = 0$ this condition is vacuous). Recall, too, that T_s is the closure of the open set of $\mathcal{Z}^{m,n}_{2,k-2s}$ where some $x_{i,j}^{(s)} \ne 0$. By Theorem 2.3, we have a one-to-one correspondence between the components of T_s and the components of

$\mathcal{Z}^{m-1,n-1}_{1,k-2s}$, a correspondence which preserves codimension in the respective spaces $\mathbf{A}^{mn(k-2s)}$ and $\mathbf{A}^{(m-1)(n-1)(k-2s)}$. Since $\mathcal{Z}^{m-1,n-1}_{1,k-2s}$ is clearly irreducible of codimension $(m-1)(n-1)(k-2s)$ (it is the origin in $\mathbf{A}^{(m-1)(n-1)(k-2s)}$), we find that each X_s is irreducible. It follows too that X_s has codimension $(m-1)(n-1)(k-2s)+mns$ in $\mathbf{A}^{mnk}$, where the extra summand mns accounts for the rows $\mathbf{u}_i^{(0)}\dots\mathbf{u}_i^{(s-1)}$ being set to zero.

As for Y_L, we have already observed in the discussion before this theorem that it is irreducible. In the case $k=2L$, Y_L is just $\mathbf{A}^{mnL}$, so it has codimension mnL in $\mathbf{A}^{2mnL}$. If $k=2L+1$, then the codimension of Y_L is mnL (corresponding to the rows $\mathbf{u}_i^{(0)},\dots,\mathbf{u}_i^{(L-1)}$ being set to zero) plus the codimension of the variety W_L. But this is known to be $(m-1)(n-1)$ (see [2] for instance; this can also be derived from Theorem 2.3).

We will now prove that the components of $\mathcal{Z}^{m,n}_{2,k}$ are as described. It is easily seen from the codimension formulas above that except when $(m,n)=(3,3)$ or $(m,n)=(3,4)$, the codimension decreases as a function of s. This shows that if $s'>s$, then X'_s (or Y_L) cannot be contained in X_s. Since the reverse containment is ruled out as $x^{(s)}_{i,j}\neq 0$ on X_s, we find that the components of $\mathcal{Z}^{m,n}_{2,k}$ are indeed as described, except in the two special cases.

To take care of these two special cases, we will use results from Section 4. (The proof works for all (m,n) pairs actually.) Using reverse induction on s, we will show that at the s-th stage, $s=L,L-1,\dots,0$, the components of Y_s are $X_s, X_{s+1},\dots,X_{L-1}$, and Y_L. We have already observed that Y_L is irreducible, so assume that $s<L$. Note that Y_s is the union of X_s and Y_{s+1}, and by induction, Y_{s+1} will have components $X_{s+1},\dots,X_{L-1}$, and Y_L. We will prove that none of these subvarieties $X_{s+1},\dots,X_{L-1}$, and Y_L can be contained in X_s. The reverse containment is ruled out as in the previous paragraph, and we will indeed find that Y_s has components $X_s, X_{s+1},\dots,X_{L-1}$, and Y_L.

We will first show that no $X_{s+\alpha}$ ($\alpha=1,\dots,L-s-1$) can be contained in X_s. For, assume to the contrary. Recall that $X_{s+\alpha}$ decomposes as $T_{s+\alpha}\times\mathbf{A}^{mn(s+\alpha)}$, where the factor $T_{s+\alpha}$ corresponds to all the entries of the rows $\mathbf{u}_i^{(l)}$, $1\le i\le m$, $l=s+\alpha,\dots,k-1-(s+\alpha)$, and the factor $\mathbf{A}^{mn(s+\alpha)}$ corresponds to all the entries of the rows $\mathbf{u}_i^{(l)}$, $1\le i\le m$, $l=k-(s+\alpha),\dots,k-1$. Recall too that $T_{s+\alpha}$ is the closure in $W_{s+\alpha}$ where some $\mathbf{u}_i^{(s+\alpha)}\neq 0$, where $W_{s+\alpha}$ has the description given earlier. The following point P is therefore in $X_{s+\alpha}$: $\mathbf{u}_1^{(s+\alpha)}=(1,0,0,\dots)$, $\mathbf{u}_2^{(k-(s+\alpha))}=(0,1,0,\dots)$, $\mathbf{u}_3^{(k-1-s)}=(0,0,1,0,\dots)$, and all other rows of all possible degrees in P are zero. (The nonzero coordinates in the row $\mathbf{u}_1^{(s+\alpha)}$ belong to $T_{s+\alpha}$ while those in $\mathbf{u}_2^{(k-(s+\alpha))}$ and $\mathbf{u}_3^{(k-1-s)}$ belong to the other factor $\mathbf{A}^{mn(s+\alpha)}$.)

Now X_s decomposes as $T_s \times \mathbf{A}^{mns}$, where the factor T_s corresponds to all the entries of $\mathbf{u}_i(l)$, $1 \le i \le m$, $l = s, \ldots, k-1-s$. Since $P \in X_s$ by assumption, an examination of the indices $s+\alpha$, $k-(s+\alpha)$, and $k-1-s$) of its nonzero rows show that P is in the subvariety $T_s \times \mathbf{O}$, where we have written $\mathbf{O}$ for the origin in $\mathbf{A}^{mns}$. Thus, the coordinates of P coming from the rows $\mathbf{u}_i(l)$, $1 \le i \le m$, $l = s, \ldots, k-1-s$ must satisfy Equations (11) in Corollary 4.4 which hold on T_s. In particular, the specific equation in (11) that holds for the coefficient of $t^{2(k-2s)-1}$, on specializing to the rows $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, reads

$$\sum_{\substack{a+b+c=2(k-2s)-1 \\ 0 \le a,b,c < k-2s}} \mathbf{u}_1^{(s+a)} \wedge \mathbf{u}_2^{(s+b)} \wedge \mathbf{u}_3^{(s+c)} = 0 \tag{12}$$

Examining the coordinates of P, and recognizing that $\mathbf{u}_2^{(k-(s+\alpha))} = \mathbf{u}_2^{(s+(k-2s-\alpha))}$ and $\mathbf{u}_3^{(k-1-s)} = \mathbf{u}_3^{(s+(k-1-2s))}$ we find that all wedge products in the equation above are already zero except possibly $\mathbf{u}_1^{(s+\alpha)} \wedge \mathbf{u}_2^{(k-(s+\alpha))} \wedge \mathbf{u}_3^{(k-1-s)}$. But by our choice of these rows, this wedge product is clearly nonzero. This shows that $X_{s+\alpha}$ is not contained in X_s.

To show that Y_L is not contained in X_s, consider first the case $k = 2L+1$. Then $Y_L \cong \mathcal{Z}_{2,1}^{m,n} \times \mathbf{A}^{mnL}$, where the factor $\mathcal{Z}_{2,1}^{m,n}$ comes from the entries of $\mathbf{u}_i(L)$, $1 \le i \le m$, and the factor $\mathbf{A}^{mnL}$ comes from the entries of $\mathbf{u}_i(l)$, $1 \le i \le m$, $l = L+1, \ldots, 2L$. Choose P to be the point with $\mathbf{u}_1^{(L)} = (1,0,0,\ldots)$, $\mathbf{u}_2^{(L+1)} = (0,1,0,\ldots)$, $\mathbf{u}_3^{(k-1-s)} = (0,0,1,0,\ldots)$, and all other rows of all possible degrees zero. (The nonzero coordinates coming from the row $\mathbf{u}_1^{(L)}$ belong to $\mathcal{Z}_{2,1}^{m,n}$ while those coming from $\mathbf{u}_2^{(L+1)}$ and $\mathbf{u}_3^{(k-1-s)}$ belong to the other factor $\mathbf{A}^{mnL}$.) Exactly as before, this point P is in Y_L, but we find that P does not satisfy Equation (11) for T_s.

When $k = 2L$, we take P to be the point with $\mathbf{u}_1^{(L)} = (1,0,0,\ldots)$, $\mathbf{u}_2^{(L)} = (0,1,0,\ldots)$, $\mathbf{u}_3^{(k-1-s)} = (0,0,1,0,\ldots)$, and all other rows of all possible degrees zero. Once again, this point P is in Y_L but does not satisfy Equation (11) for T_s. This completes the proof.

□

We get the following corollary from this:

Corollary 5.2. *The variety $\mathcal{Z}_{2,k}^{m,n}$ has $1+\lfloor k/2 \rfloor$ irreducible components. The codimension of $\mathcal{Z}_{2,k}^{m,n}$ (in $\mathbf{A}^{mnk}$) is $(m-1)(n-1)+mn\lfloor k/2 \rfloor$ if k is* *odd and* $mn\lfloor k/2\rfloor$ *if* k *is even, except in the case where* $(m,n)=(3,3)$ *or* $(m,n)=(3,4)$*. In these two special cases,* $\mathcal{Z}_{2,k}^{3,3}$ *has codimension* $4k$*, while* $\mathcal{Z}_{2,k}^{3,4}$ *has codimension* $6k$*.*

Proof. By Theorem 5.1, $\mathcal{Z}_{2,k}^{m,n}$ clearly has $1+\lfloor k/2\rfloor$ components. These have codimension $(m-1)(n-1)(k-2s)+mns$, for $s=0,1,\ldots,L=mn\lfloor k/2\rfloor$ (the case $s=L$ corresponds to the component Y_L, but is also covered by this formula). This is linear in s, and as already observed, is decreasing in s except when $(m,n)=(3,3)$ or $(m,n)=(3,4)$. It follows that except for these two cases, the component Y_L, has the least codimension. This yields the formula for the codimension of $\mathcal{Z}_{2,k}^{m,n}$ in the general case. In the two special cases, the component X_0 must have least codimension. Hence, the codimension of $\mathcal{Z}_{2,k}^{m,n}$ in these cases is given by the codimension of X_0, which is $(m-1)(n-1)k$. □

Remark 5.3. Note that when $(m,n)=(3,4)$, the codimension of the components is constant in s. Hence, in this case, all components have the same dimension.

6. Nonmaximal Minors: The General Situation

In this section, we will use Theorem 5.1 as a building block to derive some results about $\mathcal{Z}_{r,k}^{m,n}$ in general, when $r<m$. The first one is easy:

Theorem 6.1. *The variety* $\mathcal{Z}_{r,k}^{m,n}$ *in the nonmaximal case* $(r<m)$ *has at least* $1+\lfloor k/2\rfloor$ *components.*

Proof. By Theorem 2.8, $\mathcal{Z}_{r,k}^{m,n}$ has at least as many components as its subvariety Z_0, and the components of this subvariety are in one-to-one correspondence with those of $\mathcal{Z}_{r-1,k}^{m-1,n-1}$. Proceeding thus, $\mathcal{Z}_{r,k}^{m,n}$ has at least as many components as $\mathcal{Z}_{2,k}^{m-r+2,n-r+2}$, and this last variety has $1+\lfloor k/2\rfloor$ components. □

It is easy to see that there are lots of intersections between various pairs of components of $\mathcal{Z}_{2,k}^{m,n}$. From this, we get the following trivially:

Theorem 6.2. *The varieties* $\mathcal{Z}_{r,k}^{m,n}$ *when* $r<m$ *are not normal. When* $(m,n)\neq(1+r,2+r)$ *they are not pure (i.e. they have irreducible components of different dimension) and they are not Cohen-Macaulay.*

Proof. In the case of 2×2 minors, the fact that the varieties are not normal follows from the fact that there are irreducible components that, as is easy to see, intersect nontrivially. When $(m,n)\neq(3,4)$ these components are of different dimensions (see Theorem 5.1 and Remark 5.3). This implies that the corresponding varieties are not pure and not Cohen-Macaulay. In the general case, we repeatedly invoke the birational isomorphism of Theorem 2.3 between the subvariety Z_0 (which is a union of some of the components of $\mathcal{Z}_{r,k}^{m,n}$) and the variety $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbf{A}^{k(m+n-1)}$, and then reduce to the case $\mathcal{Z}_{2,k}^{m-r+2,n-r+2}$. Note that the image of the birational isomorphism is the open subset of $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbf{A}^{k(m+n-1)}$ where the free variable $x_{m,n}^{(0)} \neq 0$ (see Remark 2.4), so any intersection between components of $\mathcal{Z}_{r-1,k}^{m-1,n-1}$ of dimensions d_1 and d_2 will indeed manifest itself as an intersection between components of $\mathcal{Z}_{r,k}^{m,n}$ (in fact of the subvariety Z_0 of $\mathcal{Z}_{r,k}^{m,n}$) of dimensions $d_1 + k(m+n-1)$ and $d_2 + k(m+n-1)$. □

The fact that the variety has irreducible components of different dimension settles the question of Cohen-Macaulayness quickly. For the remaining cases $(m,n) = (1+r, 2+r)$ a more subtle analysis is needed to answer the question of Cohen-Macaulyness. Some preliminary computations communicated by the referee suggest that the varieties $\mathcal{Z}_{r,k}^{m,n}$ with $r < m$ are not Cohen-Macaulay in general.

Although in general for $r \geq 3$ (and $r < m$) it is difficult to explicitly determine the components of $\mathcal{Z}_{r,k}^{m,n}$ (the difficulty lies in determining whether some of the components of Z_1–determined by induction–lie inside some of the components of Z_0–also determined by induction: this was what the proof of Theorem 5.1 in the $r = 2$ case was all about), we have an easy reduction argument that settles the matter in the case $k < r$:

Proposition 6.3. *In the case where $k < r$, the subvariety Z_1 of $\mathcal{Z}_{r,k}^{m,n}$ is contained in Z_0. The components of $\mathcal{Z}_{r,k}^{m,n}$ and their codimensions in $\mathbf{A}^{mnk}$ in this case are hence determined completely by the components of $\mathcal{Z}_{r-1,k}^{m-1,n-1}$ and their codimensions in $\mathbf{A}^{(m-1)(n-1)k}$.*

Proof. Consider the subvariety V of $\mathcal{Z}_{r,k}^{m,n}$ defined by setting all 2×2 minors of degree zero to zero, i.e., defined by setting all $\mathbf{u}_i^{(0)} \wedge \mathbf{u}_j^{(0)} = 0$ for all $1 \leq i < j \leq m$. The equations for V are thus the standard equations for $\mathcal{Z}_{r,k}^{m,n}$ along with all 2×2 minors of degree zero. It is clear that every point on Z_1 satisfies these equations, so $Z_1 \subset V$. Since $r \geq (k-1) + 2$, every r-fold wedge product of vectors $\mathbf{u}_i^{(l)}$ of total degree at most $k-1$ must contain at least two factors of degree zero. It follows that $\mathcal{I}_{r,k}^{m,n}$ is already contained in the ideal generated by all 2×2 minors of degree zero, which is an ideal that is known classically to be prime, [2]. Hence, V is an irreducible variety, isomorphic to $\mathcal{Z}_{2,1}^{m,n}$. The components of $\mathcal{Z}_{r,k}^{m,n}$ come from either Z_1 or Z_0. Since V cannot be contained wholly in any component of Z_1 (as there are clearly points on

V where not all $x^{(0)}_{i,j}$ are zero), we find $V \subset Z_0$. It follows that $Z_1 \subset Z_0$. Thus the components of $\mathcal{Z}^{m,n}_{r,k}$ all come from Z_0, and Theorem 2.8 now finishes the proof. □

We end this section with an inductive scheme for computing the codimension of $\mathcal{Z}^{m,n}_{r,k}$ in the case $r < m$. The induction is based on r, and we will assume that for all r' with $2 \le r' < r$ and for all m, n with $r' < m \le n$, and for all $k \ge 2$, we know the codimension of $\mathcal{Z}^{m,n}_{r',k}$. (The starting point for the induction is Theorem 5.1, and the ideas here parallel the codimension computations of Theorem 5.1.)

Write $k = \lambda r + \mu$, for $\lambda \ge 1$, and $0 \le \mu < k$. (When $k < r$, we already know that the components, and their codimensions, are determined by those of $\mathcal{Z}^{m-1,n-1}_{r-1,k}$, thanks to Proposition 6.3 above.) We now have the following sequence of subvarieties (similar to the setup of Theorem 5.1):

- We will write Y_0 for our variety $\mathcal{Z}^{m,n}_{r,k}$, and X_0 for its subvariety "Z_0". Thus, X_0 is birational to $\mathcal{Z}^{m-1,n-1}_{r-1,k} \times \mathbf{A}^{k(m+n-1)}$. Write c_0 for the codimension of $\mathcal{Z}^{m-1,n-1}_{r-1,k}$ in $\mathbf{A}^{(m-1)(n-1)k}$. Then X_0 also has codimension c_0. We will assume that c_0 is known by induction.
- Proceeding, let Y_s $(s = 1, \dots, \lambda - 1)$ be the subvariety of Y_{s-1} where all $x^{(s-1)}_{i,j}$ are zero, so $Y_s \cong \mathcal{Z}^{m,n}_{r,k-rs} \times \mathbf{A}^{mns(r-1)}$. Write T_s for the subvariety "Z_0" of $\mathcal{Z}^{m,n}_{r,k-rs}$, and let $X_s = T_s \times \mathbf{A}^{mns(r-1)}$. Then X_s has codimension $c_s + smn$, where c_s is the codimension of $\mathcal{Z}^{m-1,n-1}_{r-1,k-rs}$ in $\mathbf{A}^{(m-1)(n-1)(k-rs)}$. We will assume that c_s is known.
- If $\mu = 0$, i.e., if $k = \lambda r$, then Y_λ, the subvariety of $Y_{\lambda-1}$ where all $x^{(\lambda-1)}_{i,j}$ are zero, is already an affine space of codimension $mn\lambda$ in $\mathbf{A}^{mnk}$. For convenience we will take $c_\lambda = 0$ in this case, so the codimension of Y_λ may be written for this case as $c_\lambda + \lambda mn$.
- If $\mu > 0$, then Y_λ is isomorphic to $\mathcal{Z}^{m,n}_{r,\mu} \times \mathbf{A}^{\lambda(r-1)mn}$. Since $\mu < r$, we can reduce its codimension computations to that of $\mathcal{Z}^{m-1,n-1}_{r-1,\mu}$ by Proposition 6.3, so we will assume that c_λ, the codimension of $\mathcal{Z}^{m,n}_{r,\mu}$ in $\mathbf{A}^{mn\mu}$ is known. It follows that the codimension of Y_λ is $c_\lambda + \lambda mn$.

Our result is the following:

Theorem 6.4. *The codimension of $\mathcal{Z}^{m,n}_{r,k}$ in the case $r < m$ is the minimum of the numbers $c_s + smn$, $s = 0, 1, \dots, \lambda$.*

Proof. This is clear, since $\mathcal{Z}^{m,n}_{r,k}$ is the union of the various X_s $(s = 0, 1, \dots, \lambda - 1)$ and Y_λ.

□

7. The cases $k = 2$ and $k = 3$

In this section, we determine the components of $\mathcal{Z}_{r,k}^{m,n}$ in the case $k = 2$ and $k = 3$.

Theorem 7.1. *When $k = 2$ (i.e., when we consider the "tangent bundle" to $\mathcal{Z}_{r,1}^{m,n}$), and when $r < m$, $\mathcal{Z}_{r,2}^{m,n}$ has exactly two components. One of them is the closure of any of the open sets $U_{[i_1,\dots,i_{r-1}|j_1,\dots,j_{r-1}]}$ of $\mathcal{Z}_{r,k}^{m,n}$, where the $(r-1)\times(r-1)$ minor of degree zero determined by rows $i_1,\dots,i_{r-1}$ and columns $j_1,\dots,j_{r-1}$ is nonzero, and hence also of their union. This component has codimension $2(m-r+1)(n-r+1)$. The other is the subvariety defined by setting all $(r-1)\times(r-1)$ minors of degree zero to zero, and has codimension $(m-r+2)(n-r+2)$.*

Proof. The number of components and their codimension can be obtained from repeated applications of Proposition 6.3 above, while the description of the components can be obtained by induction on r, carefully tracking prime ideals through various applications of the isomorphism of Theorem 2.3. However, we can give a more direct proof as follows:

Write U for the union of the open sets where some $(r-1)\times(r-1)$ degree zero minor of $X(0)$ is nonzero. Note that the portion of the classical degree zero variety $\mathcal{Z}_{r,1}^{m,n}$ where some $(r-1)\times(r-1)$ minor is nonzero is precisely the set of smooth points of $\mathcal{Z}_{r,1}^{m,n}$. The variety $\mathcal{Z}_{r,2}^{m,n}$ is the union of two subvarieties: one, say X, is the closure of U, and the other, say Y is the subvariety where all $(r-1)\times(r-1)$ degree zero minors of $X(0)$ are zero. It is easy to see that U is irreducible of the stated codimension, since the fibers over any point of $\mathcal{Z}_{r,1}^{m,n}$ where some $(r-1)\times(r-1)$ is nonzero are all linear spaces of the same dimension. Hence X is irreducible of the stated codimension. It is also easy to see that the Jacobian matrix defining tangent spaces to classical variety $\mathcal{Z}_{r,1}^{m,n}$ is zero when all $(r-1)\times(r-1)$ minors are zero, so indeed, the tangent spaces at such points are simply copies of $\mathbf{A}^{mn}$. Since the base space $\mathcal{Z}_{r-1,1}^{m,n}$ is irreducible, Y is irreducible as well, and it has the stated codimension. It is clear that X cannot be contained in Y as U is nonempty. As for the other direction, consider the point in $\mathbf{A}^{2mn}$ with $\mathbf{u}_i^{(0)} = (0,\dots,1,\dots,0)$ where the 1 is in the i-th slot, for $i = 1,\dots,r-2$, $\mathbf{u}_i^{(1)} = (0,\dots,1,\dots,0)$ where the 1 is in the i-th slot, for $i = r-1, r, r+1$, and all other $\mathbf{u}_i^{(l)}$ equal to the zero vector. Then this point is in Y, but is not in X, since it clearly does not satisfy Equation (10) of Corollary 4.4. □

When $k = 3$, we need to consider only the $r = 3$ situation. For, the $r = 2$ case is covered by Theorem 5.1, while the components for $r > 3$ sequentially reduce to the components for the $r = 3$ case by Proposition 6.3. We determine these components for all but a finite set of (m, n) pairs:

Theorem 7.2. *For all $r \geq 3$, and for all (m, n) pairs with $3 < m \leq n$ except possibly the pairs $(1+r, 1+r), (1+r, 2+r), (1+r, 3+r), (1+r, 4+r)$, and $(2+r, 2+r)$, the variety $\mathcal{Z}_{r,3}^{m,n}$ has exactly three components. The first, say X, has codimension $(m-r+3)(n-r+3)$ in $\mathbf{A}^{3mn}$. The second, say Y, has codimension $(m-r+1)(n-r+1)+(m-r+2)(n-r+2)$, and the third, say Z, has codimension $3(m-r+1)(n-r+1)$.*

Proof. We start first with $\mathcal{Z}_{3,3}^{m,n}$. What we have called X here is the subvariety Z_1 of Theorem 2.8, it is just $\mathbf{A}^{2mn}$. The subvarieties Y and Z of this theorem correspond to the components of $\mathcal{Z}_{2,3}^{m-1,n-1}$, which by Theorem 2.8 are in bijection with the components of $\mathcal{Z}_{3,3}^{m,n}$ coming from the closure of where some $x_{i,j}^{(0)} \neq 0$. Note that this bijection preserves codimensions, so from Theorem 5.1, Y and Z indeed have the stated codimensions. It remains only to check that these are indeed components. By Theorems 2.8 and 5.1, Y and Z are components of $\mathcal{Z}_{3,3}^{m,n}$, so it only remains to show that X is not contained in either one of these two. But it is easy to see that except possibly for the stated exceptional values of (m, n), we have $cd(X) < cd(Y) < cd(Z)$, so X cannot be contained in Y or Z. (Here $cd(X)$ is the codimension of the variety X.)

For the general case of $r \geq 3$, we invoke Proposition 6.3 repeatedly, along with Theorem 2.8 to reduce to the case of $\mathcal{Z}_{3,3}^{m-r+3,n-r+3}$. □

Remark 7.3. The components for $k = 4$ will be covered in [7] below.

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Dept. of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Dept. of Mathematics, California State University Northridge, Northridge CA 91330, U.S.A.

E-mail address: tomaz.kosir@fmf.uni-lj.si

E-mail address: al.sethuraman@csun.edu