COMMUTING PAIRS AND TRIPLES OF MATRICES AND RELATED VARIETIES

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Abstract. In this note, we show that the set of all commuting d-tuples of commuting $n \times n$ matrices that are contained is an n-dimensional commutative algebra is a closed set, and therefore, Gerstenhaber's theorem on commuting pairs of matrices is a consequence of the irreducibility of the variety of commuting pairs. We show that the variety of commuting triples of 4×4 matrices is irreducible. We also study the variety of n-dimensional commutative subalgebras of $M_n(F)$, and show that it is irreducible of dimension $n^2 - n$ for $n \leq 4$, but reducible, of dimension greater than $n^2 - n$ for $n \geq 7$.

1. Introduction

Let F be an algebraically closed field. We are interested in the affine variety of commuting d-tuples of elements in the ring $M_n(F)$ of $n \times n$ matrices over F. Denote this variety by $\mathcal{C}(d, n, F) = \mathcal{C}(d, n)$.

The first major result about this variety was due to Motzkin and Taussky [6] in a fundamental paper. They proved that:

Theorem 1. *$\mathcal{C}(2, n)$ is irreducible.*

Gerstenhaber [2] reproved this result and obtained other results. In particular, the following result was proved in [2]:

Theorem 2. *If A, B are commuting elements of $M_n(F)$, then A and B are contained in an n-dimensional commutative subalgebra of $M_n(F)$.*

See [7] for a nice proof of the result. Note two immediate corollaries of this:

Theorem 3. *Let A and B be commuting elements of $M_n(F)$ and let $\mathcal{A}$ be the algebra generated by A and B.*

1. $\dim \mathcal{A} \leq n$; *and*
2. $\dim \operatorname{Cent}(\mathcal{A}) \geq n$.

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There were several proofs of Part (1) of Theorem 3 above given in [5], [10] and [1]. It was pointed out in [3] that Theorem 3 follows very easily from Theorem 1 (both these conditions are closed conditions on $\mathcal{C}(2,n)$ and they hold on the nonempty open subset where A has distinct eigenvalues – by Theorem 1, this subset is dense, whence the results).

In this note, we set up a certain incidence correspondence in a suitable product of projective varieties, and then project onto one component to find that the pairs of commuting matrices that are contained in some commutative subalgebra of $M_n(F)$ of dimension n is a closed subset of $\mathcal{C}(2,n)$. Since the nonempty open subset of commuting pairs where the first matrix has distinct eigenvalues is contained in this closed subset, all of $\mathcal{C}(2,n)$ must be in this set. We thus find that Theorem 2 is also a consequence of Theorem 1.

Projecting from the correspondence onto a different component, we find that the set of commutative subalgebras of $M_n(F)$ of dimension n has the natural structure of an algebraic variety: it is a closed set in the Grassmanian variety of n-dimensional subspaces of n^2 dimensional space. We study this variety and show that for $n = 2$, 3, and 4, it is an irreducible variety of dimension $n^2 - n$, and that for general n, it has a component of dimension $n^2 - n$. We show that for $n \geq 7$, this variety is not irreducible.

We also study $\mathcal{C}(3,n)$. Gerstenhaber ([2]) had proved that $\mathcal{C}(d,n)$ is reducible for $d \geq 4$ and $n \geq 4$ and had asked if $\mathcal{C}(3,n)$ is irreducible. The first author had proved ([3]) that $\mathcal{C}(3,n)$ is reducible for $n \geq 32$, but irreducible for $n \leq 3$. Using results from [8], we show that $\mathcal{C}(3,4)$ is also irreducible.

2. The Incidence Correspondence.

We use elementary algebraic geometry to study the incidence relation between d-tuples of commuting matrices and commutative subalgebras of $M_n(F)$ of dimension n.

Write $\mathcal{C}_{\mathbf{P}}(d,n)$ for the projectivization of the affine variety of commuting d-tuples of $n \times n$ matrices. (Note the commutativity of matrices is a homogenous condition on the entries of the matrices.) We will denote the equivalence class of a commuting d-tuple $A_1, \ldots, A_d$ by $(A_1 : \cdots : A_d)$. Recall that the set of n-dimensional subspaces of an n^2 dimensional space has the natural structure of a projective variety via the Plucker embedding which associates to any n-dimensional space W with basis $v_1, \ldots, v_n$, the point $[W] = v_1 \wedge \cdots \wedge v_n$ in $\mathbf{P}(\wedge^n(V))$ (see [4] for instance; this embedding is independent of the basis $v_1, \ldots, v_n$ chosen). Denote this variety by $\mathbf{G}(n,n^2)$. In the product variety

$\mathcal{C}_{\mathbf{P}}(d,n)\times\mathbf{G}(n,n^2)$, consider the set X of all points $((A_1:\cdots:A_d),[W])$ subject to the following conditions:

1. Each A_i is in the subspace W.
2. W is closed under multiplication.
3. The multiplication in W is commutative.
4. The identity matrix I_n is in W.

Each of these is a polynomial condition on $\mathcal{C}_{\mathbf{P}}(d,n)\times\mathbf{G}(n,n^2)$. For instance, the first and fourth conditions simply read $A_i \wedge W = 0$ and $I_n \wedge W = 0$ in $\wedge^{n+1}V$, and of course, these wedge products can be read off the coordinates of A_i and the coordinates of the point $[W]$. As for the second and third conditions, note that there exists an open cover $\{U_i\}$ of $\mathbf{G}(n,n^2)$ such that on each U_i, a basis for W can be read off the coordinates of the point $[W]$. (This can be gleaned from the discussion in [4, Page 65]. For instance, a point [W] for which the first coordinate is nonzero can be represented as the row space of a unique matrix of the form $(I_n \mid M)$, where M is an $n\times(n^2-n)$ matrix. Note that the entries of M differ by just a sign from suitable $n\times n$ subdeterminants of the matrix $(I_n \mid M)$. Thus, the coordinates of a basis for W differ by just a sign from from suitable Plucker coordinates of $[W]$.) If b_k $(k=1,\ldots,n)$ are the basis vectors represented in terms of the coordinates of a point $[W]$ in some open set of this cover, then the second condition translates on this open set to $(b_kb_l)\wedge W = 0$ in $\mathbf{P}(\wedge^{n+1}(V))$, for all k and all l, while the third condition translates on this open set to $b_kb_l = b_lb_k$ for all k and all l. It follows that X is locally closed in $\mathcal{C}_{\mathbf{P}}(d,n)\times\mathbf{G}(n,n^2)$, and hence closed ([9, Chapter 1, §4.2]).

Since X is a closed projective variety, the projections $\pi_1 : X \to \mathcal{C}_{\mathbf{P}}(d,n)$ and $\pi_2 : X \to \mathbf{G}(n,n^2)$ have closed images ([9, Chapter 1, §5.2]). We thus have the following:

Proposition 4.
1. *The set of commutative d-tuples that are contained in some n-dimensional commutative subalgebra of $M_n(F)$ of dimension n is a closed subset of $\mathcal{C}(d,n)$.*
2. *The set of commutative subalgebras of $M_n(F)$ of dimension n is a closed subset of $\mathbf{G}(n,n^2)$.*

Proof.
1. The set of commutative d-tuples that are contained in some n-dimensional commutative subalgebra of $M_n(F)$ of dimension n is just the cone over $\pi_1(X)$.
2. This was essentially proved above (see the discussions on Conditions 2, 3, and 4). Note that the set of commutative subalgebras of $M_n(F)$ of dimension n is precisely $\pi_2(X)$.

□

Remark 5. It is clear that similar considerations apply to subalgebras of $M_n(F)$ of any dimension d, satisfying any set of polynomial identities.

3. Commuting d-tuples of Matrices.

Let us write $\mathcal{Z}(d,n)$ for the set of commutative d-tuples that are contained in some n-dimensional commutative subalgebra of $M_n(F)$ of dimension n. It now follows very easily from Proposition 4 that Theorem 2 is a consequence of Theorem 1: the nonempty open subset of $\mathcal{C}(2,n)$ where the first matrix has distinct eigenvalues is contained in $\mathcal{Z}(2,n)$; since $\mathcal{C}(2,n)$ is irreducible, this set is dense in $\mathcal{C}(2,n)$, and therefore, since $\mathcal{Z}(2,n)$ is closed, all of $\mathcal{C}(2,n)$ must be contained in $\mathcal{Z}(d,n)$.

More generally, let $X_1(d,n)$ be the nonempty open subset of $\mathcal{C}(d,n)$ where the first matrix is nonderogatory. (Recall that a matrix A in $M_n(F)$ is nonderogatory if $[F[A]:F]=n$, equivalently, if $\mathrm{Cent}(A)=F[A]$. This condition, of course, includes matrices with distinct eigenvalues.) Let $X(d,n)$ denote the closure of $X_1(d,n)$. Let $Y(d,n)$ denote the the subset of $\mathcal{C}(d,n)$ where the last $d-1$ matrices commute with a nonderogatory matrix. The proof of Theorem 1, as found, for instance, in [3], consists of showing that $X_1(2,n)$ is irreducible, and thus that $X(2,n)$ is irreducible, and then showing that $Y(2,n)$ is contained in $X(2,n)$. Since every matrix commutes with a nonderogatory matrix, we find $Y(2,n)=\mathcal{C}(2,n)$, and thus, $\mathcal{C}(2,n)=X(2,n)$ is irreducible.

The same proof techniques apply to show the following:

Proposition 6. *$X_1(d,n)$, and hence $X(d,n)$, is irreducible of dimension $n^2+(d-1)n$, and $Y(d,n)\subseteq X(d,n)$.*

Proof. If $(A_1,\ldots,A_d)$ is in $X_1(d,n)$, then there exist uniquely determined polynomials f_2, $\ldots$, f_d of degree at most $n-1$ such that $A_i=f_i(A_1)$ $(i=2,\ldots,d)$. Thus, $X_1(d,n)$ is parametrized by the nonderogatory matrices and $d-1$ copies of the space of polynomials of degree at most $n-1$, from which the irreducibility and dimension follow. Also, if A_2, $\ldots$, A_d commute with a nonderogatory matrix B, then the line $((1-t)A_1+tB, A_2,\ldots,A_d)$ is contained in $\mathcal{C}(d,n)$, and a nonempty subset of the line is contained in the open set $X_1(d,n)$. Hence, the entire line must be in $X(d,n)$, and in particular, so must the point $(A_1,\ldots,A_d)$. □

The following is a consequence of Proposition 4 above.

Theorem 7. *Let A_1, $\ldots$, A_d be in $X(d,n)$. Let $\mathcal{A}=F[A_1,\ldots,A_d]$. Then*

1. *$\mathcal{A}$ is contained in an n-dimensional commutative subalgebra of $M_n(F)$.*
2. *The dimension of $\mathcal{A}$ is at most n, while the dimension of the centralizer of $\mathcal{A}$ is at least n.*

Proof. If A_1 is nonderogatory, then $\mathcal{A} = F[A_1]$. It follows that $X_1(d,n)$ is contained in $\mathcal{Z}(d,n)$, so the closure $X(d,n)$ is also contained in $\mathcal{Z}(d,n)$. Statement (1) now follows. Statement (2) is an immediate consequence of Statement (1).

□

We now turn our attention to $\mathcal{C}(3,4)$, the variety of commuting triples of 4×4 matrices. As explained in the Introduction, it was shown by the first author ([3]) that $\mathcal{C}(3,n)$ is reducible for $n \geq 32$ but irreducible for $n \leq 3$. Using a result from [8], we prove the following:

Theorem 8. *$\mathcal{C}(3,4)$ is an irreducible variety.*

Proof. As proved in [8, Theorem 15], the set of all commuting triples (A,B,C) in which some two of A, B, and C commute with a 2-regular matrix is contained in the closure of $X_1(3,n)$. (Recall that a matrix is i-regular if each eigenspace is at most i-dimensional.) Since this closure (denoted $X(3,n)$ above and $\overline{U_1}$ in [8]) is irreducible, it is sufficient to show that any two commuting 4×4 matrices commute with a 2-regular matrix to be able to conclude that $\mathcal{C}(3,4)$ is irreducible.

Let A and B be two commuting 4×4 matrices. We may assume that $V \cong F^4$ is an indecomposable $F[A,B]$ module, since otherwise, the problem reduces to showing that any two commuting matrices of size at most 3 commute with a 2-regular matrix, and here, the problem is trivial. Thus, A and B have only one eigenvalue, and subtracting this off, we may assume that A and B are both nilpotent. Moreover, we may assume that A and B are not 2-regular, and that neither is a scalar. Thus, both A and B are nilpotent matrices that are 3-regular but not 2-regular, so the Jordan form for each is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix}$$

Thus, each has a 3-dimensional nullspace and a 1-dimensional image contained in the nullspace. The common kernel of A and B, by a dimension count, must be at least 2-dimensional.

It follows very easily from the Jordan form of the matrices that any 1-dimensional subspace of V that is invariant under A must be contained in the kernel of A (and similarly for B as well). Since BV is 1-dimensional and $A(BV) = B(AV) \subset BV$, we find that BV is contained in the kernel of A. Thus, $A(BV) = B(AV) = 0$, and of course, $A(AV) = B(BV) = 0$, so both AV and BV are 1-dimensional subspaces contained in the common kernel of A and B.

Let W be a 2-dimensional subspace of V contained in the common kernel of A and B that contains both AV and BV. Extend a basis w_1, w_2 of W to a basis w_1, w_2, w_3, w_4 of V. In this basis, consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(the i-th column represents $M(w_i)$). This is a 2-regular matrix, and $AM = MA = 0$, and $BM = MB = 0$. This proves the theorem. □

We point out an immediate corollary:

Corollary 9. *If A, B, and C are any three commuting 4×4 matrices, the algebra $F[A, B, C]$ is contained in some commutative subalgebra of $M_4(F)$ of dimension 4. Hence, $F[A, B, C]$ has dimension at most 4, and the centralizer of $F[A, B, C]$ has dimension at least 4.*

Proof. With the irreducibility of $C(3, 4)$ at hand, this is now standard: the closure of $X_1(3, 4)$ must be all of $C(3, 4)$, and Theorem 7 now applies. □

Remark 10. The irreducibility of $C(3, n)$ is still open for $5 \le n \le 31$.

4. The variety of n-dimensional commutative subalgebras of $M_n(F)$.

By Proposition 4, the set of commutative subalgebras of $M_n(F)$ of dimension n has the structure of a closed set in $\mathbf{G}(n, n^2)$. Let A_n denote this variety.

Our first result is a lower bound for the dimension of A_n:

Proposition 11. *For all $n \ge 2$, the closure in A_n of the set of algebras generated by nonderogatory matrices is an irreducible variety of dimension $n^2 - n$. In particular, the dimension of A_n is at least $n^2 - n$.*

Proof. Let $U \subseteq \mathbf{A}^{n^2}$ be the open set of non-derogatory matrices. We define a map $\phi\colon U \to A_n$ by sending the matrix A to the algebra generated by A; explicitly, ϕ sends A to $1 \wedge A \wedge \cdots \wedge A^{n-1}$. The fiber over $F[A]$ is the open set of nonderogatory matrices in the algebra

$F[A]$, and is thus of dimension n. Since U is irreducible of dimension n^2, the closure of $\phi(U)$ must be irreducible of dimension $n^2 - n$. □

For small values of n, we can sharpen Proposition 11 further.

Theorem 12. *For $n \leq 4$, A_n is an irreducible variety of dimension n^2-n. In particular, for $n \leq 4$, the algebras generated by nonderogatory matrices are dense in A_n.*

Proof. By [3] and Theorem 8 above, the set of commuting triples of $n \times n$ matrices is an irreducible variety for $n = 2$, 3, and 4. For $n \leq 4$, let $U \subseteq C(3,n)$ be the open set of triples (A, B, C) such that the F-algebra generated by A, B, and C has dimension exactly n. For each subset T of $\{(i,j,k) \mid 0 \leq i,j,k \leq n-1\}$ of cardinality n, let $U_T \subseteq U$ be the open set of triples (A, B, C) such that the matrices $\{A^iB^jC^k \mid (i,j,k) \in T\}$ are linearly independent. (So, U is simply the union over all T of the U_T.) We define $\phi_T \colon U_T \to A_n$ by sending the triple (A, B, C) to $\bigwedge_{(i,j,k)\in T} A^iB^jC^k$. The morphisms ϕ_T on U_T patch together to give a morphism $\phi \colon U \to A_n$ that sends a triple (A, B, C) to the algebra $F[A, B, C]$. Since $n \leq 4$, every algebra in A_n needs at most 3 generators, so ϕ is surjective. For any point $F[A, B, C]$ in A_n, the fiber over the point is the intersection of $F[A, B, C] \times F[A, B, C] \times F[A, B, C]$ with U, an open set of dimension $3n$. Since U is irreducible of dimension $n^2 + 2n$ (this follows from the irreducibility of $C(3, n)$ for $n \leq 4$ and Proposition 6), we find that $\phi(U) = A_n$ is irreducible of dimension $n^2 - n$. The last statement of the theorem follows from this and Proposition 11. □

Before we investigate A_n for higher n, we need an observation:

Lemma 13. *For any positive integer d, the set of algebras in A_n that can be generated by d elements is an open set of A_n.*

Proof. As described in the discussions before Proposition 4, $\mathbf{G}(n, n^2)$ has an open cover $\{U_i\}$ such that in each U_i, there is an explicit formula for determining a set of basis vectors of a subspace W from the coordinates of the point $[W] \in U$. Fix one such open set U, and let b_j $(j = 1, \ldots, n)$ be a basis of W represented as a function of the coordinates of $[W]$. Let $x_{i,j}$ $(1 \leq i \leq d,\ 1 \leq j \leq n)$ be a new set of indeterminates, and let $B_i = \sum_j x_{i,j} b_j$ $(1 \leq i \leq d)$ be d generic matrices in W. Form the $n^2 \times n^d$ matrix whose columns are the entries of $B_1^{i_1} \ldots B_d^{i_d}$ $(0 \leq i_1, \ldots, i_d \leq n-1)$. The various $n \times n$ subdeterminants of this matrix are polynomials in the variables $x_{i,j}$, and the condition that there be some set of generators of W of cardinality d is precisely the condition that not all these polynomials be identically zero, that is, that not all coefficients of all these polynomials vanish. This is an open set condition on U, and the union of such open sets over all the U_i is precisely the set of d generated algebras. □

Corollary 14. *For each $n \geq 2$, A_n has a component of dimension $n^2 - n$. In particular, if A_n is irreducible, it must have dimension $n^2 - n$.*

Proof. Lemma 13 above shows that the one-generated algebras form an open set in A_n, and Proposition 11 shows that this open set must be irreducible. Hence, this set must be contained wholly within some component of A_n, and being open in that component, must be dense. It follows that that component must have dimension $n^2 - n$. The last statement is clear. □

Theorem 15. *For $n \geq 7$, A_n is not irreducible.*

Proof. Write $n = 2k$ for n even, and $n = 2k + 1$ for n odd. Write V for F^n, and let $v_1, \ldots, v_n$ be a basis for V. Let X be the variety $\{[W] \in \mathbf{G}(n-1, n^2) \mid f \cdot g = 0 \ \forall\ f, g \in W \text{ and } \dim(\mathrm{F} - \mathrm{span}\{\mathrm{image}(f) \mid f \in W\}) \leq k\}$. (Here, we identify F^{n^2} with $\mathrm{End}_F(V)$, so the product $f \cdot g$ is a product of matrices. One shows that X is a variety by writing out a basis for W in terms of its Plucker coordinates—see the discussions preceding Proposition 4—and observing that both conditions on $[W]$ are polynomial conditions.) Let U be the open set of this variety consisting of all $[W]$ that satisfy the additional requirement that $\dim(\mathrm{F} - \mathrm{span}\{\mathrm{image}(f) \mid f \in W\})$ is exactly k. For each $[W] \in U$, let $S_W = \mathrm{F} - \mathrm{span}\{\mathrm{image}(f) \mid f \in W\}$. We have a map $\phi\colon U \to \mathbf{G}(k, n)$ given by $[W] \mapsto [S_W]$. We claim that this map is a surjective morphism.

On an open set V of $\mathbf{G}(n-1, n^2)$, let $f_1, \ldots, f_{n-1}$ be a basis for a subspace W belonging to U, written in terms of the Plucker coordinates of $[W]$. For each subset T of $\{(i,j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n\}$ of cardinality k, let V_T be the open set of V where the vectors $f_i(v_j)$, $(i,j) \in T$, are linearly independent. Then $\phi([W]) = \bigwedge_{(i,j)\in T} f_i(v_j)$ on V_T, so ϕ is a morphism. Now given any subspace Y of V of dimension k, note that the set of $f \in \mathrm{End}_F(V)$ such that $f(V) \subseteq Y$, and $f(Y) = 0$ can be written in a suitable basis as block 2×2 matrices, with the $(1,1)$, $(2,1)$ and $(2,2)$ blocks being zero. (The product of any two of these matrices is zero.) It is now trivial to find a subspace of dimension $n-1$ of these matrices the span of whose images is all of Y. The surjectivity of ϕ is now clear.

The fiber over any $[Y]$ in $\mathbf{G}(k,n)$ is the set $\{[W] \in \mathbf{G}(n-1, n^2) \mid f(V) \subseteq Y,\ f(Y) = 0 \ \forall\ f \in W \text{ and } \mathrm{F} - \mathrm{span} f(V) = Y, f \in W\}$. By writing matrices in a suitable basis as in the paragraph above, one sees that this is an open set of $\mathbf{G}(n-1,k^2)$ when n is even, and of $\mathbf{G}(n-1,k(k+1))$ when n is odd. If U' is an irreducible component of U such that $\phi(U')$ is dense in $\mathbf{G}(k,n)$ (such a component must exist, as $\mathbf{G}(k,n)$ is irreducible), we find by ([9, Chapter 1, §6.3, Theorem 7]) that $\dim(U')$ is the sum of $\dim(\mathbf{G}(k,n))$ and the dimension of the fibers. Thus, in the even case, we find $\dim(U) \geq \dim(U') = k^2 + (2k-1)(k-1)^2$, and in the odd case, $\dim(U) \geq \dim(U') = k(k+1) + 2k(k^2-k)$.

But we have a map $U \to A_n$ that sends $[W]$ to the algebra $W + F \cdot I_n$. It is easy to see that this map is a morphism, and using the fact that any $f \in W$ is nilpotent, it is easy to see that the map is injective. Hence, $\dim(A_n) \geq \dim(U)$. For $n \geq 8$ in the even case, and $n \geq 7$ in the odd case, we find that $\dim(A_n)$ is greater than $n^2 - n$. By Corollary 14, A_n cannot be irreducible for $n \geq 7$. □

Example 16. For $n = 2$, we can determine the equations of A_2 quite explicitly.

Let $\mathcal{A}$ be a 2-dimensional subalgebra of $M_2(F)$ (such an algebra is necessarily commutative). Let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} x & a \\ b & y \end{pmatrix}$ be a basis. Then $\mathcal{A}$ is represented by the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ x & a & b & y \end{pmatrix},$$

in $\mathbf{G}(2,4) \subseteq \mathbf{P}^5$. Thus, A is represented by the point $(a : b : y - x : 0 : -a : -b)$. Letting z_i $(i = 1, \ldots, 6)$ stand for the i-th homogenous coordinate on $\mathbf{P}^5$, the subspace corresponding to $\mathcal{A}$ lives in the linear subspace of $\mathbf{P}^5$ given by $z_1 + z_5 = 0$, $z_2 + z_6 = 0$, and $z_4 = 0$. Conversely, given a point in this linear subspace of $\mathbf{P}^5$, note that not all of z_1, z_2 and z_3 can be zero, so I_2 and any matrix of the form $M = \begin{pmatrix} u & z_1 \\ z_2 & z_3 + u \end{pmatrix}$ (for arbitrary $u \in F$) are linearly independent and hence generate a 2-dimensional algebra. The point corresponding to this algebra is precisely $(z_1 : z_2 : z_3 : 0 : -z_1 : -z_2)$.

We thus find: The set of 2-dimensional subalgebras of $M_2(F)$ corresponds to the 2-dimensional linear subspace of $\mathbf{P}^5$ given by the equations $z_1 + z_5 = 0$, $z_2 + z_6 = 0$, and $z_4 = 0$.

Since a matrix of the form $\begin{pmatrix} u & z_1 \\ z_2 & z_3 + u \end{pmatrix}$ (for arbitrary $u \in F$) has distinct eigenvalues *iff* $z_3^2 + 4z_1z_2 \neq 0$, we also find: The open subset of this linear subspace given by $z_3^2 + 4z_1z_2 \neq 0$ corresponds to algebras generated by matrices with distinct eigenvalues, while the subvariety

$z_3^2 + 4z_1z_2 = 0$ corresponds to algebras generated by conjugates of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

One can show numerically that A_3 and A_4 are not linear subvarieties of $\mathbf{P}(\wedge^3(F^3))$ and $\mathbf{P}(\wedge^4(F^4))$ by showing that the linear span of the set of algebras generated by nonderogatory matrices has dimension higher than (respectively) $3^2 - 3$ and $4^2 - 4$.

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