

# SHELLABILITY OF A COMPLEX ASSOCIATED TO THE FIRST ORDER JET SCHEME OF A DETERMINANTAL VARIETY

BOYAN JONOV

ABSTRACT. In the two papers [4] and [5], Košir and Sethuraman proved that  $\mathcal{Z}_{2,2}^{m,n}$ , the first order jet scheme over the variety  $\mathcal{Z}_2^{m,n}$  of  $m \times n$  matrices of rank at most 1, has two components: a principal component  $Z_0$  which is the closure of the jets over the nonsingular points of  $\mathcal{Z}_2^{m,n}$ , and a second component supported over the origin that is just affine space. They determined a Groebner basis for the ideal  $I_0$  of  $Z_0$ ; the lead terms of this ideal are square free monomials. We show in this paper that the Stanley-Reisner simplicial complex attached to the lead terms is shellable. This allows us to conclude that the coordinate ring of the principal component of  $\mathcal{Z}_{2,2}^{m,n}$  is Cohen-Macaulay.

## 1. INTRODUCTION

Let  $F$  be an algebraically closed field and  $\mathbf{A}_F^k$  the affine space of dimension  $k$  over  $F$ . By a *variety* in  $\mathbf{A}_F^k$  we will mean the zero set of a collection of polynomials over  $F$  in  $k$  variables; in particular, our varieties are not assumed irreducible. In [4], Košir and Sethuraman had studied the following varieties  $\mathcal{Z}_{r,k}^{m,n}$  defined as follows: Consider the truncated polynomial ring  $F[t]/(t^k)$  ( $k = 1, 2, 3, \dots$ ), and let  $X(t) = (f_{i,j}(t))_{i,j}$  be the generic  $m \times n$  ( $m \leq n$ ) matrix over this ring; thus, the  $(i, j)$  entry of  $X$  is of the form  $f_{i,j}(t) = x_{i,j}^{(0)} + x_{i,j}^{(1)}t + \dots + x_{i,j}^{(k-1)}t^{k-1}$ , where for various  $i, j$  and  $l$  the  $x_{i,j}^{(l)}$  are variables. Let  $\mathcal{I}_{r,k}^{m,n}$  be the ideal of  $R = F[x_{i,j}^{(l)} \mid 1 \leq i \leq m, 1 \leq j \leq n, 0 \leq l < k]$  generated by the coefficients of powers of  $t$  in each  $r \times r$  minor of the generic matrix  $X(t)$ , and define  $\mathcal{Z}_{r,k}^{m,n} \subseteq \mathbf{A}_F^{nmk}$  to be the zero set of  $\mathcal{I}_{r,k}^{m,n}$ . (When  $k = 1$ , of course, we simply recover the classical determinantal varieties.) These varieties are the closed points of the  $(k - 1)$ -th order jet schemes of the classical determinantal varieties. In particular, Košir and Sethuraman showed that  $\mathcal{Z}_{2,2}^{m,n}$ , which we may view loosely as the “tangent bundle” over the variety of  $m \times n$  matrices of rank at most 1, has two components when  $m \geq 3$ : one,  $Z_0$ , is the set of all tangents at the nonsingular points of the base variety  $\mathcal{Z}_2^{m,n}$ , while the other,  $Z_1$ , is supported over the origin in  $\mathcal{Z}_2^{m,n}$ , and is simply the affine space  $\mathbf{A}_F^{mn}$ . Thus,  $Z_0$  is the interesting component of  $\mathcal{Z}_{2,2}^{m,n}$ , and we refer to it as the *principal component*. (When  $m = 2$ ,  $\mathcal{Z}_{2,2}^{m,n}$  is irreducible.) Recall that a Groebner basis  $G$  of an ideal of polynomials  $S$  has the property that the ideal given by the leading terms of polynomials

in  $S$  is itself generated by the leading terms of the basis  $G$ . In [5], Košir and Sethuraman determine a Groebner basis for the ideal  $I_0$  of the principal component with respect to a suitable term ordering. (When  $m = 2$ ,  $I_0$  is just  $I_{2,2}^{2,n}$ .) The lead terms of the basis are square free, and hence we may consider the Stanley-Reisner simplicial complex  $\Delta_{LT(I_0)}$  attached to the lead terms ideal  $LT(I_0)$ . (By this, we mean the simplicial complex on a suitably labeled vertex set, whose Stanley-Reisner ideal—see [6, Chap. 1] or [1, Chap. 5] for instance—is the ideal  $LT(I_0)$ .)

The goal of this paper is to prove that  $\Delta_{LT(I_0)}$  is shellable. We will enumerate all the facets of  $\Delta_{LT(I_0)}$  and we will describe an explicit ordering of the facets that establishes the shellability. By standard results, this shows that the coordinate ring of the principal component is Cohen-Macaulay. (This result has been independently been obtained in [7] as well, using their geometric technique for computing syzygies.) In addition, we will exploit our understanding of  $\Delta_{LT(I_0)}$  to compute the Hilbert series of the coordinate ring of  $Z_0$ .

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## 2. THE IDEAL $LT(I_0)$

We consider the truncated polynomial ring  $F[t]/(t^2)$  and let  $X(t) = (f_{i,j}(t))_{i,j}$  be the generic  $m \times n$ , ( $m \leq n$ ) matrix over that ring. Thus, the  $(i, j)$  entry of  $X$  is of the form  $f_{i,j}(t) = x_{i,j} + y_{i,j}(t)$ , where  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $x_{i,j}, y_{i,j}$  are variables. Let  $I_{2,2}^{m,n}$  be the ideal of  $R = F[x_{i,j}, y_{i,j}]$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ , generated by the coefficients of powers of  $t$  in each  $2 \times 2$  minors of the generic matrix  $X(t)$ . If we expand the  $2 \times 2$  minors of  $X(t)$ , we see that the ideal  $I_{2,2}^{m,n}$  is generated by the family of polynomials  $\delta_{[i,j],[k,l]}$  and  $\epsilon_{[i,j],[k,l]}$ , where

$$\delta_{[i,j],[k,l]} = \det \begin{pmatrix} x_{i,k} & x_{i,l} \\ x_{j,k} & x_{j,l} \end{pmatrix} = x_{i,k}x_{j,l} - x_{i,l}x_{j,k}, \quad (1)$$

and

$$\epsilon_{[i,k][j,l]} = x_{i,k}y_{j,l} + y_{i,k}x_{j,l} - x_{i,l}y_{j,k} - y_{i,l}x_{j,k}. \quad (2)$$

For example, if

$$X(t) = \begin{pmatrix} x_{1,1} + y_{1,1}t & x_{1,2} + y_{1,2}t & x_{1,3} + y_{1,3}t \\ x_{2,1} + y_{2,1}t & x_{2,2} + y_{2,2}t & x_{2,3} + y_{2,3}t \end{pmatrix} \quad (3)$$

then  $I_{2,2}^{2,3} = \langle x_{1,1}x_{2,2} - x_{1,2}x_{2,1}, x_{1,1}x_{2,3} - x_{1,3}x_{2,1}, x_{1,2}x_{2,3} - x_{1,3}x_{2,2}, x_{1,1}y_{2,2} + y_{1,1}x_{2,2} - x_{1,2}y_{2,1} - y_{1,2}x_{2,1}, x_{1,1}y_{2,3} + y_{1,1}x_{2,3} - x_{1,3}y_{2,1} - y_{1,3}x_{2,1}, x_{1,2}y_{2,3} + y_{1,2}x_{2,3} - x_{1,3}y_{2,2} - y_{1,3}x_{2,2} \rangle$ .

Let the monomials in  $R$  be ordered by the graded reverse lexicographical order using the following scheme:  $y_{1,1} > y_{1,2} > \cdots > y_{1,n} > y_{2,1} > \cdots >$

$y_{2,n} > \cdots > y_{m,n} > x_{1,1} > x_{1,2} > \cdots > x_{1,n} > x_{2,1} > \cdots > x_{2,n} > \cdots > x_{m,n}$ . Also, consider the following notation:

$$\rho_{[i,j,k][p,q,r]} = \det \begin{pmatrix} y_{i,p} & y_{i,q} & y_{i,r} \\ y_{j,p} & y_{j,q} & y_{j,r} \\ x_{k,p} & x_{k,q} & x_{k,r} \end{pmatrix}, \quad (4)$$

$$\lambda_{[i,j,k][p,q,r]} = \det \begin{pmatrix} y_{i,p} & y_{i,q} & x_{i,r} \\ y_{j,p} & y_{j,q} & x_{j,r} \\ y_{k,p} & y_{k,q} & x_{k,r} \end{pmatrix}, \quad (5)$$

$$\psi_{[i,j,k][p,q,r]} = \det \begin{pmatrix} y_{i,p} & y_{i,q} & y_{i,r} \\ y_{j,p} & y_{j,q} & y_{j,r} \\ y_{k,p} & y_{k,q} & y_{k,r} \end{pmatrix}. \quad (6)$$

In [5] Košir and Sethuraman provide a Groebner basis for  $I_{2,2}^{m,n}$ :

**Theorem 2.1.** ([5, Th. 2.1]) *A Groebner basis for  $I = I_{2,2}^{m,n}$  ( $2 \leq m \leq n$ ) with respect to the grevlex ordering described above consists of the five families of polynomials  $\Delta = \{\delta_{[i,j][k,l]} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $\mathcal{E} = \{\epsilon_{[i,j][k,l]} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $\mathcal{R} = \{\rho_{[i,j,k][p,q,r]} \mid 1 \leq i < j \leq k \leq m, 1 \leq p < q < r \leq n\}$ ,  $\Lambda = \{\lambda_{[i,j,k][p,q,r]} \mid 1 \leq i < j < k \leq m, 1 \leq p < q \leq r \leq n\}$ , and  $\Xi = \{x_{s,t}\psi_{[i,j,k][p,q,r]} \mid 1 \leq s \leq m, 1 \leq t \leq n, 1 \leq i < j < k \leq m, 1 \leq p < q < r \leq n\}$ .*

As described in Section 1, we focus on the principal component  $Z_0$ , with ideal  $I_0$ .

**Theorem 2.2.** ([5, Th. 2.4, Prop. 3.3, and Rem. 2.2]) *Assume  $m \geq 3$ . Then  $I_0$  is defined by the polynomials of the families  $\Delta$ ,  $\mathcal{E}$ , and  $\Psi = \{\psi_{[i,j,k][p,q,r]} \mid 1 \leq i < j < k \leq m, 1 \leq p < q < r \leq n\}$ . A Groebner basis of  $I_0$  with respect to the grevlex ordering above consists of the polynomials in the families  $\Delta$ ,  $\mathcal{E}$ ,  $\mathcal{R}$ ,  $\Lambda$ , and  $\Psi$ . When  $m = 2$ , a Groebner basis consists of all  $\delta_{[1,2][k,l]}$  and all  $\epsilon_{[1,2][k,l]}$  with  $1 \leq k < l \leq n$ , and all  $\rho_{[1,2,2][p,q,r]}$  with  $1 \leq p < q < r \leq n$ .*

Now, having the Groebner basis  $G$  of  $I_0$ , we can easily describe the leading term ideal of  $I_0$ :  $LT(I_0) = \langle lm(g); g \in G \rangle$ . With the monomial ordering specified above, we can see that  $LT(I_0)$  is generated by the following family of monomials:

**Corollary 2.3.** (*Generators of  $LT(I_0)$* ) *The following families of polynomials generate  $LT(I_0)$ :  $A = \{x_{i,l}x_{j,k} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $B = \{x_{i,k}y_{j,l} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $C = \{x_{k,p}y_{j,q}y_{i,r} \mid 1 \leq i < j \leq k \leq m, 1 \leq p < q < r \leq n\}$ ,  $D = \{x_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m, 1 \leq p < q \leq r \leq n\}$ , and  $E = \{y_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m, 1 \leq p < q < r \leq n\}$ .*

Since  $LT(I_0)$  is generated by squarefree monomials we can construct the Stanley Reisner complex of  $LT(I_0)$ : this is the simplicial complex on vertices  $(x_{i,j}, y_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n)$  whose corresponding Stanley Reisner ideal

(see [6, Chap. 1] or [1, Chap. 5]) is  $LT(I_0)$ . It is defined by the relation  $x_{i_1, j_1} \cdots x_{i_s, j_s} y'_{i'_1, j'_1} \cdots y'_{i'_t, j'_t} \mid 1 \leq s \leq m, 1 \leq t \leq n$  is a face of  $\Delta_{LT(I_0)}$  if  $x_{i_1, j_1} \cdots x_{i_s, j_s} y'_{i'_1, j'_1} \cdots y'_{i'_t, j'_t}$ , as a monomial, does not belong to  $LT(I_0)$ . Our goal, as stated in Section 1, is to prove that  $\Delta_{LT(I_0)}$  is shellable.

It would be helpful in what follows to visualize the structure of the monomials in the families  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ . For this, see Figures 1, 2, 3, 4, and 5. In this paper, we will visualize a monomial as being positioned in a matrix, where each variable of the monomial is located in the matrix's entry corresponding to the index of the variable.

### 3. DESCRIBING THE FACETS OF $\Delta_{LT(I_0)}$

In this section, we will enumerate all facets of  $\Delta_{LT(I_0)}$ . First, some notation: we will denote a facet  $F$  of the simplicial complex  $\Delta_{LT(I_0)}$  by  $F = F_x F_y$ , where  $F_x$  is a string composed of vertices  $x_{i,j}$ 's and  $F_y$  is a string composed of vertices  $y_{i,j}$ 's. We will view each of  $F_x$  and  $F_y$  as both strings of vertices or monomials, depending on the context. Note that  $F_x F_y \in \Delta_{LT(I_0)}$  if, as a monomial,  $F_x F_y$  does not belong in the ideal  $LT(I_0)$ . Also, since  $F_x F_y$  is a monomial,  $F_x F_y$  does not belong in the ideal  $LT(I_0)$  if and only if  $F_x F_y$  is not divisible by the generators of  $LT(I_0)$ .

We will start by showing a relation between the facets of  $\Delta_{LT(I_0)}$  and those of the corresponding simplicial complexes arising from classical determinantal varieties. We refer to the excellent survey paper of Bruns and Conca [8]. In this paper, the authors consider the facets of  $\Delta_t$ : the Stanley-Reisner complex attached to the ideal  $in(I_t)$ , which is generated by the leading terms of the  $t \times t$  minors of the generic  $m \times n$  matrix  $(x_{i,j})$ . The order they use is one in which the leading term of a minor is the main diagonal, and it is known that the leading terms of the  $t \times t$  minors generate the ideal of leading terms of  $I_t$ . The key result for us is [8, Prop. 6.4], where they enumerate the facets of  $\Delta_t$ . This is a purely combinatorial result that enumerates the maximal subsets of  $V = \{1, \dots, m\} \times \{1, \dots, n\}$  that intersect any  $t$ -subset of  $V$  arising from the diagonal of some  $t \times t$  submatrix of  $(x_{i,j})$  in at most  $t - 1$  places, and can be applied by symmetry to enumerate the maximal subsets of  $V$  that intersect any  $t$ -subset of  $V$  arising from the antidiagonal of some  $t \times t$  submatrix of  $(x_{i,j})$  in at most  $t - 1$  places. We quote this result as:

**Proposition 3.1.** ([8, Prop. 6.4]). *Let  $I_t$  be the ideal of  $F[\{x_{i,j}\}]$  generated the  $t \times t$  minors of the generic  $m \times n$  matrix  $(x_{i,j})$ . Write  $LT(I_t)$  for the ideal generated by the lead terms of the  $t \times t$  minors with respect to the graded reverse lexicographical order  $x_{1,1} > x_{1,2} > \cdots > x_{1,n} > x_{2,1} > \cdots > x_{2,n} > \cdots > x_{m,n}$ . Write  $\Delta'_t$  for the Stanley Reisner complex of  $LT(I_t)$ . Then the facets of  $\Delta'_t$  are given by all families of non-intersecting paths from  $(1, 1), (2, 1), \dots, (t - 1, 1)$  to  $(m, n), (m, n - 1), \dots, (m, n - t + 2)$ .*

Here, a path from  $(a, b)$  to  $(c, d)$ , given  $a \leq c$  and  $b \leq d$ , is a sequence of vertices starting at  $x_{a,b}$  and ending at  $x_{c,d}$  where each vertex in the sequence is either one step to the right or one step down from the previous vertex. A nonintersecting path of the kind described in the last line of the proposition above is a union of paths from  $(i, 1)$  to  $(m, n - i + 1)$  whose pairwise intersection is empty. (It is known that for the graded reverse lexicographic order as well, the leading terms of the  $t \times t$  minors generate the ideal of leading terms of  $I_t$ .)

We observe that the monomials in  $A$  correspond to the generators of  $I_2$  above, and the monomials in  $E$  correspond to the generators of  $I_3$  above. We therefore have the following:

**Lemma 3.2.**  *$F_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$  and  $F_y$  is a subset of a pair of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$*

*Proof.* Since  $F_x$  is not divisible by a monomial in  $A$ , i.e.  $F_x$  is not in  $\text{in}(I_2)$ , it follows that  $F_x$  is a subset of a facet of  $\Delta_2$ . Hence, by Proposition 3.1 above, it follows that  $F_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$ . Similarly, since  $F_y$  is not divisible by a monomial in  $E$ , i.e.  $F_y$  is not in  $\text{in}(I_3)$ , it follows that  $F_y$  is a subset of a facet of  $\Delta_3$ . Hence, by Proposition 3.1 above, it follows that  $F_y$  is a subset of a pair of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$ .  $\square$

We will continue by showing that for each facet  $F = F_x F_y$  of  $\Delta_{LT(I_0)}$ ,  $F_x$  is a non-empty string that contains at least two  $x$ -vertices.

**Proposition 3.3.** *Let  $F$  be a facet of the simplicial complex  $\Delta_{LT(I_0)}$ . Then  $F$  must contain  $x_{m,n}$ .*

*Proof.* Suppose that  $F$  does not contain  $x_{m,n}$ . Then, consider the string  $x_{m,n}F$ . Using the fact that the facet  $F$  is by itself not divisible by any of the monomials in  $A, B, C, D$ , or  $E$ , it is easy to check that  $x_{m,n}F$  also can not be divisible by a monomial in any of those families. So,  $x_{m,n}F \in \Delta_{LT(I_0)}$  and maximality of  $F$  implies that  $x_{m,n}$  must already be in  $F$ .  $\square$

**Proposition 3.4.** *Let  $F$  be a facet of the simplicial complex  $\Delta_{LT(I_0)}$ . Then  $F$  must contain at least two  $x$ -vertices.*

*Proof.* From Proposition 3.3, we know that  $F$  must have at least one  $x$ -vertex, namely  $x_{m,n}$ . Suppose that it is the only one. Consider then  $x_{m-1,n}F$ . Note that  $F$  is not divisible by any of the monomials in  $A, B, C, D$ , or  $E$ . We can easily check then that  $x_{m-1,n}F$  is not divisible by any of the monomials in  $A, B, C, D$ , or  $E$  (for  $A$  we use the fact that  $x_{m,j}, j < n$  is not in  $F$  by assumption). So,  $x_{m-1,n}F \in \Delta_{LT(I_0)}$  and maximality of  $F$  implies that  $x_{m-1,n}$  must already be in  $F$ , a contradiction.  $\square$

*Notation:* By  $\mu(F_x)$  we will denote the  $x$ -vertex that is furthest to north and furthest to the west of all other  $x$ -vertices in  $F_x$ . Thus,  $\mu(F_x) = x_{a,b}$

such that  $a \leq c$  and  $b \leq d$  for all  $x_{c,d}$  in  $F_x$ . Notice that  $\mu(F_x) \neq x_{m,n}$  (Proposition 3.4).

The next proposition deals with the  $F_y$  part of a facet  $F$  and, in particular, the propositions lists some of the  $y$ -vertices that must be present in a given facet.

**Proposition 3.5.** *Let  $F = F_x F_y$  be a facet of the simplicial complex  $\Delta_{LT(I_0)}$  with  $\mu(F_x) = x_{i,j}$ . Then  $F$  must contain  $y_{i,n}$  and  $y_{m,j}$ .*

*Proof.* Suppose that  $F$  does not contain  $y_{i,n}$ . Then, consider the string  $y_{i,n}F$ . Note that  $F$  is a facet and thus it is not divisible by a monomial in  $A$ ,  $B$ ,  $C$ ,  $D$ , or  $E$ . We will show that  $y_{i,n}F$  is also not divisible by a monomial in  $A$ ,  $B$ ,  $C$ ,  $D$ , or  $E$  which would imply that  $y_{i,n}F$  is a face in  $\Delta_{LT(I_0)}$  containing  $F$  - a result contradicting the maximality of the facet  $F$ . Thus,  $F$  must contain  $y_{i,n}$ .

Obviously,  $y_{i,n}F$  can not be divisible by a monomial in  $A$ . Monomials in  $A$  involve only  $x$ -variables and  $y_{i,n}F$  being divisible by a monomial in  $A$  implies that  $F$  must itself be divisible by that monomial - a contradiction.

Suppose that  $y_{i,n}F$  is divisible by a monomial in  $B$ . Then, there must be some  $x_{a,b}$  in  $F$  (see Figure 6) such that  $x_{a,b}$  is to the north-west of  $y_{i,n}$ . If  $x_{a,b}$  is to the right of column  $j$ , then  $x_{i,j}x_{a,b}$  is in  $F$  and it is a monomial in  $A$  - a contradiction. If  $x_{a,b}$  is in column  $j$  or to the left, then we have a contradiction with the fact that  $\mu(F_x) = x_{i,j}$ . So,  $y_{i,n}F$  is not divisible by a monomial in  $B$ .

Suppose that  $y_{i,n}F$  is divisible by a monomial in  $C$ . Then, there must be some  $x_{c,d}$  and  $y_{s,t}$  in  $F$  such that  $x_{c,d}y_{s,t}y_{i,n}$  is in  $C$  (recall Figure 3). Using the fact that  $\mu(F_x) = x_{i,j}$  and the location of  $y_{i,n}$ , it follows that  $y_{s,t}$  can only be in the region to the south-east of  $x_{i,j}$  (see Figure 7). But then,  $x_{i,j}y_{s,t}$  is in  $B$  and in  $F$  - a contradiction. So,  $y_{i,n}F$  is not divisible by a monomial in  $C$ . Suppose that  $y_{i,n}F$  is divisible by a monomial in  $D$ . Then, there must be some  $x_{e,f}$  and  $y_{k,l}$  in  $F$  such that  $x_{e,f}y_{k,l}y_{i,n}$  is in  $D$  (recall Figure 4). This can only happen if  $x_{e,f}$  is in column  $n$ , above  $y_{i,n}$  (see Figure 8). However, if  $x_{i,j}$  is not in column  $n$  then  $x_{i,j}x_{e,f}$  is in  $F$  and it is a monomial in  $A$  - a contradiction. Else if  $x_{i,j}$  is in column  $n$ , then  $x_{e,f}$  in  $F$  contradicts the fact that  $\mu(F_x) = x_{i,j}$ . So, in either case,  $y_{i,n}F$  is not divisible by a monomial in  $D$ .

Finally, suppose that  $y_{i,n}F$  is divisible by a monomial in  $E$ . Then there must be some  $y_{a,b}$  and  $y_{c,d}$  in  $F$  such that  $y_{i,n}y_{a,b}y_{c,d}$  is in  $E$  (recall Figure 5). In particular, it must be the case that, say,  $y_{c,d}$  is to the south-west of  $y_{a,b}$  which, in turn, is to the south-west of  $y_{i,n}$  (see Figure 9). Now, if  $y_{a,b}$  is to the south-east of  $x_{i,j}$ , then  $x_{i,j}y_{c,d}$  is in  $B$  and in  $F$  - a contradiction. If  $y_{a,b}$  is to the south or south-west of  $x_{i,j}$ , then  $x_{i,j}y_{c,d}y_{a,b}$  is in  $D$  and  $F$  - a contradiction. So,  $y_{i,n}F$  is not divisible by a monomial in  $E$ .

So,  $y_{i,n}F$  is not divisible by a monomial in  $A$ ,  $B$ ,  $C$ ,  $D$ , or  $E$  which implies, as argued above, that  $F$  must contain  $y_{i,n}$ . We can similarly show that  $F$  must contain  $y_{m,j}$  as well.  $\square$

Now we are ready to describe the structure of all facets of  $\Delta_{LT(I_0)}$ . The following notation will be useful in the next theorem: for a given facet  $F = F_x F_y$  with  $\mu(F_x) = x_{i,j}$ , consider the following partition of the  $y$ -vertices based on the index  $(i, j)$ :  $R_1 = \{y_{s,t} \mid s \leq i, j < t\}$ ,  $R_2 = \{y_{s,t} \mid s \leq i, t \leq j\}$ ,  $R_3 = \{y_{s,t} \mid i < s, j < t\}$ ,  $R_4 = \{y_{s,t} \mid i < s, t \leq j\}$  (see Figure 10).

**Theorem 3.6.** *Let  $F = F_x F_y$  be a facet of the simplicial complex  $\Delta_{LT(I_0)}$  with  $\mu(F_x) = x_{i,j}$ . Then  $F_x$  is a path from  $x_{i,j}$  to  $x_{m,n}$  and  $F_y$  is a family of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$ .*

*Proof.* Slight changes in the arguments in the following proof would occur based on the location of the  $\mu(F_x) = x_{i,j}$  vertex and so the proof would be accordingly divided in three cases. In each case, we will first show that  $F = F_x F_y$  as described in the theorem is indeed a valid facet of  $\Delta_{LT(I_0)}$ . Then we will argue that any facet of  $\Delta_{LT(I_0)}$  must have that form.

Case 1:  $\mu(F_x) = x_{i,j}$  is neither in row  $m$  nor in column  $n$ . Let  $F = F_x F_y$  with  $F_x$  a path from  $x_{i,j}$  to  $x_{m,n}$  and  $F_y$  a family of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$  be given (see Figure 12 and Figure 13). We will first show that  $F$  is a facet of  $\Delta_{LT(I_0)}$ , i.e.  $F$  is not divisible by monomials in  $A, B, C, D$ , or  $E$  and  $F$  is maximal with respect to inclusion.

Obviously,  $F_x$  (see Figure 12) does not contain a monomial in  $A$  (recall Figure 1, or see Proposition 3.1) and so  $F = F_x F_y$  is not divisible by a monomial in  $A$ . If  $F$  were divisible by a monomial in  $B$ , i.e.  $x_{a,b} y_{c,d}$  is in  $F$  and it is a monomial in  $B$  (recall Figure 2), then notice that  $y_{c,d}$  would have to be in  $R_4$  - impossible (see Figure 13). So,  $F$  is not divisible by a monomial in  $B$ . Suppose that  $F$  is divisible by a monomial in  $C$ , i.e.  $x_{a,b} y_{c,d} y_{e,f}$  is in  $F$  and it is a monomial in  $C$  (recall Figure 3). Then notice that we will have two  $y$ -vertices, one to the south-west of the other, that both must be in  $R_1 \cup R_4$  - a contradiction (see Figure 13). So,  $F$  is not divisible by a monomial in  $C$ . Suppose that  $F$  is divisible by a monomial in  $D$ , i.e.  $x_{a,b} y_{c,d} y_{e,f}$  is in  $F$  and it is a monomial in  $D$  (recall Figure 4). Then notice that in this case we will have two  $y$ -vertices, one to the south-west of the other, that both must be in  $R_3 \cup R_4$  - a contradiction (see Figure 13). So,  $F$  is not divisible by a monomial in  $D$ . Finally, from Figure 13, it is obvious that  $F$  can not be divisible by a monomial in  $E$  (recall Figure 5).

Next, we will show that  $F = F_x F_y$  is maximal with respect to inclusion by arguing that any vertex attached to  $F$  would make the resulting monomial divisible by some monomial in  $A, B, C, D$ , or  $E$  (i.e. that resulting monomial can not be a face in  $\Delta_{LT(I_0)}$ ). Recall that  $F_x$  is a subset of facet of  $\Delta_2$ , i.e. a path from  $x_{1,1}$  to  $x_{m,n}$ . So, if we attach a vertex  $x_{a,b}$  to  $F_x$ , it has to be to the north, west or north-west of  $x_{i,j}$ . But notice that in this case either  $x_{a,b} y_{m,j}$  or  $x_{a,b} y_{i,n}$ , or both, would be a monomial in  $B$ . So, no  $x$ -vertex can be attached to  $F$ . Recall also that  $F_y$  is a subset of facet of  $\Delta_3$ , i.e. of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$ . So, if we attach a vertex  $y_{c,d}$  to  $F_y$ , it has to be in  $R_4$ . But this means that  $x_{i,j} y_{c,d}$  would be a monomial in  $B$ . So, no  $y$ -vertex can be attached to  $F$ . Since

no  $x$  or  $y$ -vertex can be attached to  $F$ , then  $F$  is maximal with respect to inclusion.

Finally, we will show that any facet  $f = f_x f_y$  of  $\Delta_{LT(I_0)}$  with  $\mu(f_x) = x_{i,j}$  must of the form described in the theorem. Since  $f_x$  must be a subset of a facet of  $\Delta_2$ , i.e. a path from  $x_{1,1}$  to  $x_{m,n}$ , and since  $\mu(f_x) = x_{i,j}$ , then it follows that  $f_x$  must actually be a subset of a path from  $x_{i,j}$  to  $x_{m,n}$ . Next,  $f_y$  must be a subset of facet of  $\Delta_3$ , i.e. of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$ . By Proposition 3.5, it follows that  $f_y$  must also contain  $y_{i,n}$  and  $y_{m,j}$ . Furthermore,  $f_y$  can not contain  $y$ -vertices in  $R_4$ , because  $x_{i,j}$  and any vertex in that region is a monomial in  $B$ . So, those last three conditions imply that  $f_y$  must be a subset of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$ . Finally, notice that  $f = f_x f_y$  is actually a subset of a some facet  $F$  as described in the theorem. Maximality of  $f$  implies that it actually has to be one of those facets  $F$ .

Case 2:  $\mu(F_x) = x_{i,j}$  is in row  $m$ , i.e.  $i = m$  (see Figure 14 and Figure 15). As in Case 1, we can easily check that  $F$  is not divisible by monomials in  $A, B, C, D$ , or  $E$ .

Next, we will show that  $F$  is maximal. With the same arguments as in Case 1, we see that if some vertex  $x_{a,b}$  is to be attached to  $F_x$ , it has to be to the north, west or north-west of  $x_{i,j}$ . However, if  $x_{a,b}$  is to the north or north-west of  $x_{i,j}$ , then  $x_{a,b} y_{m,n}$  would be a monomial in  $B$ . If  $x_{a,b}$  is to the west of  $x_{i,j}$ , then  $x_{a,b} y_{m,j}$  and some  $y$ -variable that is in the upper path of  $F_y$  and in  $R_1$  would produce a monomial in  $C$ . So, no  $x$ -vertex can be attached to  $F$ . Using the arguments from Case 1 again, we see that if we attach a vertex  $y_{c,d}$  to  $F_y$ , then this vertex has to be in  $R_1$  ( $R_1$  is the only region in which  $F_y$  can be completed to a facet of  $\Delta_3$ ). However, in this case,  $x_{i,j} y_{c,d}$  and some  $y$ -variable that is in the upper path of  $F_y$  and in  $R_1$  would again produce a monomial in  $C$ . So, no  $y$ -vertex can be attached to  $F$  as well.

We have to finally show that any facet  $f = f_x f_y$  of  $\Delta_{LT(I_0)}$  with  $\mu(f_x) = x_{m,j}$  must of the form described in the theorem. As in Case 1,  $f_x$  must be a subset of a facet of  $\Delta_2$ , i.e. a path from  $x_{1,1}$  to  $x_{m,n}$ , and since  $\mu(f_x) = x_{m,j}$ , then it follows that  $f_x$  must actually be a subset of a path from  $x_{m,j}$  to  $x_{m,n}$ . As for  $f_y$ , it must be a subset of some facet of  $\Delta_3$ , which is a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$  and those two paths must also contain  $y_{m,n}$  and  $y_{m,j}$  (Proposition 3.5). Furthermore, because of  $x_{m,j}$ , there should be no  $y$ -vertices in  $f_y \cap R_1$  such that one is to the south-west of the other (otherwise  $x_{m,j}$  and those two  $y$ -vertices would produce a monomial in  $C$ ). Therefore,  $f_y \cap R_1$  must actually be a subset of a facet in  $\Delta_2$  on vertex set  $R_1$ , i.e.  $f_y \cap R_1$  must be a subset of a some path in  $R_1$  ending at  $y_{m,n}$ . Putting together all the information about  $f_y$ , we see that  $f_y$  must be a subset of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,j}$ . So, again by maximality argument,  $f = f_x f_y$  must be a facet of the form as described in the theorem.

Case 3:  $\mu(F_x) = x_{i,j}$  is in column  $n$ , i.e.  $j$ . This case is handled similarly to case Case 1 and Case 2.  $\square$

Knowing the structure of a facet  $F = F_x F_y$  of the simplicial complex  $\Delta_{LT(I_0)}$ , we can easily count the number of vertices that  $F$  is composed of, so we can determine  $\dim F = |F| - 1$ . In particular, we see that the dimension of any facet  $F$  is  $2(m+n) - 3$ . Notice that the dimension of  $F$  depends only on the constants  $m$  and  $n$ . Thus, we can conclude that all facets of the simplicial complex  $\Delta_{LT(I_0)}$  have the same dimension, i.e.  $\Delta_{LT(I_0)}$  is a pure simplicial complex of dimension  $2(m+n) - 3$ .

**Corollary 3.7.** *The dimension of  $R/I_0$  is  $2(m+n) - 2$ .*

Theorem 3.6 also allows to determine the total number of facets in  $\Delta_{LT(I_0)}$ . Thus, we can determine the multiplicity of  $R/I_0$  as well.

**Corollary 3.8.** *The multiplicity of  $R/I_0$  is given by*

$$\sum_{(i,j), (i,j) \neq (m,n)} \binom{m+n-i-j}{m-i} \det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix} \quad (7)$$

*Proof.* The number of paths from  $(i, j)$  to  $(m, n)$  is  $\binom{m+n-i-j}{m-i}$ , while the number of non-intersecting paths from  $(1, 1)$  to  $(i, n)$  and from  $(2, 1)$  to  $(m, j)$  is given by (see [10, §2.2])

$$\det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix}$$

$\square$

*Remark 3.9.* Professor B.A. Sethuraman had conjectured that the expression for the multiplicity of  $R/I_0$  above should simplify to  $\binom{n+m-2}{m-1}^2$ . This has since been proven by Professor Sudhir Ghorpade ([2]).

#### 4. SHELLABILITY OF $\Delta_{LT(I_0)}$

The main goal of this section is to prove that our simplicial complex  $\Delta_{LT(I_0)}$  is shellable. Recall the following definition of shellability:

*Definition 4.1.* A simplicial complex  $\Delta$  is *shellable* if it is pure and if its facets can be given a total order, say  $F_1, F_2, \dots, F_e$ , so that the following condition holds: for all  $i$  and  $j$  with  $1 \leq j < i \leq e$  there exists  $v \in F_i \setminus F_j$  and an index  $k$ ,  $1 \leq k < i$ , such that  $F_i \setminus F_k = \{v\}$ . A total order of the facets satisfying this conditions is called *shelling* of  $\Delta$ .

**Theorem 4.2.** *The simplicial complex  $\Delta_{LT(I_0)}$  is shellable.*

*Proof.* Note that at the end of the previous section we have argued that  $\Delta_{LT(I_0)}$  is pure. We will proceed by first giving a partial order to the facets of  $\Delta_{LT(I_0)}$ . Let  $P = P_x P_y$  and  $Q = Q_x Q_y$  be two facets of  $\Delta_{LT(I_0)}$ . If  $\mu(P_x)$  is in a row below  $\mu(Q_x)$ , we set  $P < Q$  (see Figure 16). If  $\mu(P_x)$  and  $\mu(Q_x)$  are in the same row, but  $\mu(P_x)$  is to the right of  $\mu(Q_x)$ , we set  $P < Q$  (see Figure 17). If  $\mu(P_x) = \mu(Q_x)$  but  $P_x$  is to the right of  $Q_x$  as one goes from  $\mu(P_x)$  to  $x_{m,n}$ , then  $P < Q$  (see Figure 18). If  $P_x = Q_x$  and the upper  $y$ -path of  $P_y$  goes to the right of the upper  $y$ -path of  $Q_y$ , we set  $P < Q$ . Finally, if  $P_x = Q_x$  and the lower  $y$ -path of  $P_y$  goes to the right of the upper  $y$ -path of  $Q_y$ , we set  $P < Q$ . Now we arbitrarily extend this partial order on the facets of  $\Delta_{LT(I_0)}$  to a total order.

Now we will prove that the selected total order is indeed a shelling of  $\Delta_{LT(I_0)}$ . Let  $P = P_x P_y$  and  $Q = Q_x Q_y$  be two facets of  $\Delta_{LT(I_0)}$  such that  $P < Q$ . Our goal is to find  $v \in Q \setminus P$  and a facet  $R < Q$  such that  $Q \setminus R = \{v\}$ . Suppose that  $\mu(P_x) \neq \mu(Q_x)$ . Notice  $P$  can not contain  $\mu(Q_x) = x_{i,j}$  (otherwise  $P < Q$  is contradicted). Take  $v = x_{i,j}$ . Take  $R = R_x R_y$  to be the following:  $R_x = Q_x \setminus x_{i,j}$  (see Figure 19) and  $R_y = Q_y y_{m,j+1}$  if  $\mu(R_x) = x_{i,j+1}$  (see Figure 20) or  $R_y = Q_y y_{i+1,n}$  if  $\mu(R_x) = x_{i+1,j}$ .

Next, suppose that  $\mu(P_x) = \mu(Q_x)$ , but  $P_x \neq Q_x$ . Then, there must be a right turn  $H = x_{a,b}$  in  $Q_x$  that is not in  $P_x$ . So, in this case take  $v = H = x_{a,b}$  and  $R = R_x R_y$  where  $R_x = Q_x$  with  $H = x_{a,b}$  replaced by  $x_{a+1,b-1}$  and  $R_y = Q_y$  (see Figure 21).

Next, suppose that  $P_x = Q_x$  and the upper  $y$ -paths of the two facets are different. Notice that the upper path of  $Q_y$  can not be strictly on the right of the upper path of  $P_y$  (otherwise  $P < Q$  is contradicted). So, there must be a right turn  $H = y_{c,d}$  of the upper path of  $Q_y$  strictly on the left of the upper path of  $P_y$ . Thus,  $H = y_{c,d}$  can not be in  $P_y$ . So, take  $v = y_{c,d}$ . If  $y_{c+1,d-1}$  is not in the lower path of  $Q_y$ , let  $R = R_x R_y$  be the following facet:  $R_x = Q_x$  and  $R_y = Q_y$  with  $y_{c,d}$  replaced by  $y_{c+1,d-1}$ . If  $y_{c+1,d-1}$  is in the lower path of  $Q_y$  (see Figure 22), then notice that  $y_{c+1,d-1}$  must be a right turn as well. Then take  $R = R_x R_y$  to be the following facet:  $R_x = Q_x$  and  $R_y$  is obtained from  $Q_y$  by replacing  $y_{c,d}$  with  $y_{c+1,d-1}$  in the upper path of  $Q_y$  and by replacing  $y_{c+1,d-1}$  with  $y_{c+2,d-2}$  in the lower path of  $Q_y$ .

Finally, suppose that  $P_x = Q_x$ , the upper  $y$ -paths of the two facets are the same, but the lower  $y$ -paths are different. Similarly as in the previous paragraph, we see that there must be a right turn  $H = y_{e,f}$  of the lower path of  $Q_y$  strictly on the left of the lower path of  $P_y$ . Notice that  $H = y_{e,f}$  can not be in the upper path of  $P_y$  because it the same as the upper path of  $Q_y$ . So, take  $v = y_{e,f}$ . Let  $R = R_x R_y$  be the facet:  $R_x = Q_x$  and  $R_y = Q_y$  with  $y_{e,f}$  replaced by  $y_{e+1,f-1}$ .

□

The following result, which was a primary reason for studying the shellability of  $\Delta_{LT(I_0)}$  is now immediate (this has been independently obtained in [7] as well using very different techniques):

**Corollary 4.3.** *The coordinate ring of  $Z_0$ , the principal component of  $\mathcal{Z}_{2,2}^{m,n}$ , is Cohen-Macaulay.*

*Proof.* The coordinate ring of  $Z_0$  is the ring  $R/I_0$ . By standard results, the ring  $R/I_0$  is Cohen-Macaulay if the ring  $R/LT(I_0)$  is Cohen-Macaulay (see [6, Corollary 8.31]. By construction,  $R/LT(I_0)$  is precisely the Stanley-Reisner ring associated to  $\Delta_{LT(I_0)}$ , and since  $\Delta_{LT(I_0)}$  is shellable,  $R/LT(I_0)$  will necessarily be Cohen-Macaulay (see [1, Theorem 5.1.13]).  $\square$

### 5. HILBERT SERIES OF THE COORDINATE RING OF $Z_0$

The Hilbert series of  $R/I_0$  coincides with that of  $R/LT(I_0)$ , and the shellability of  $\Delta_{LT(I_0)}$  can be used to determine the Hilbert series of  $R/LT(I_0)$ . Suppose that  $F_1, F_2, \dots, F_e$  is a shelling of  $\Delta_{LT(I_0)}$ . Consider the following notation:  $c(F_t) = \{v \in F_t : \text{there exists } s < t \text{ such that } F_t \setminus F_s = \{v\}\}$ . Then the Hilbert series of  $R/LT(I_0)$  is of the form  $h(z)/(1-z)^d$  with  $h(z) = \sum h_k z^k \in \mathbb{Z}[z]$ ,  $h_0 = 1$  and  $h_k = |\{t \in \{1, \dots, e\} : |c(F_t)| = k\}|$  (see [8, Theorem 6.3]),  $d = \dim \Delta_{LT(I_0)} + 1 = 2(m+n) - 2$  and  $k \leq d$  (see [1, Lemma 5.1.8]). For a facet  $F = F_x F_y$  of  $\Delta_{LT(I_0)}$ , it is easy to see that  $v \in c(F)$  implies that either  $v = \mu(F_x)$  (as long as  $\mu(F_x) \neq x_{m,n-1}$ ) or  $v$  is a right turn of  $F$  (by saying that  $v$  is a right turn of  $F$ , we mean that  $v$  is a right turn of either  $F_x$ , the upper, or the lower path of  $F_y$ ). The only case when a right turn of a facet  $F$  that is not in  $c(F)$  occurs when  $F$  has a right turn in the upper path of  $F_y$  that immediately to the north-east of  $y_{m,j}$  (see Figure 11). Let  $C_{i,j}^k$  denote the number of facets  $F = F_x F_y$  of  $\Delta_{LT(I_0)}$  such that  $F$  has exactly  $k$  - right turns that are in  $c(F)$  and  $\mu(F_x) = x_{i,j}$ . With this notation, we see that the coefficient  $h_k$  of the Hilbert series of  $R/LT(I_0)$  can be expressed in the following way:

$$h_k = C_{m,n-1}^k + \sum_{(i,j) \neq (m,n-1)} C_{i,j}^{k-1} \tag{8}$$

At this point, we see that an essential part of determining the coefficients  $h_j$  is finding the number of non-intersecting lattice paths containing certain fixed number of right turns. The enumeration of non-intersecting lattice paths with a given total number of turns is provided by [9, Theorem 2] (see Equation (10) ahead). So far in this paper, the facets of  $\Delta_{LT(I_0)}$  were represented as a string of vertices where each vertex was linked to an entry of an  $m \times n$  matrix (recall Figure 12 and Figure 13). To accommodate for Krattenthaler's result it will be useful to make two changes of our representation of the facets of  $\Delta_{LT(I_0)}$ . First, the matrix in which we picture our facets standing will be rotated 90 deg to the left (thus, the indexing of the vertices of the facets would be Cartesian co-ordinates compatible). Second, after the rotation has taken place, we will consider the upper path of  $L_y$  to be the path starting at  $y_{1,2}$  and ending at  $y_{i,n}$  and the lower path of  $L_y$  to be the path starting at  $y_{1,1}$  and ending at  $y_{m,j}$  (see Figure 23 and Figure 24).

Now, let  $\mathcal{P}(\{(a, b)\}, \{(c, d)\})_r$  denote the number of paths of with  $r$  right turns such that those paths start at  $x_{a,b}$  and end at  $x_{c,d}$ . We easily see that  $\mathcal{P}(\{(a, b)\}, \{(c, d)\})_r = \binom{c-a}{r} \binom{d-b}{r}$ . Let  $\mathcal{P}(\{(a, b), (c, d)\}, \{(e, f), (g, h)\})_r$  denote the number of nonintersecting paths with  $r$  right turns such that the first path starts at  $y_{a,b}$  and ends at  $y_{e,f}$ , while the second path starts at  $y_{c,d}$  and ends at  $y_{g,h}$ . With this notation, we see that when  $i \neq m$

$$C_{i,j}^{k'} = \sum_{k_1+k_2=k'} \mathcal{P}(\{(i, j)\}, \{(m, n)\})_{k_1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(i, n), (m, j)\})_{k_2}$$

This equality holds because for a facet  $F = F_x F_y$  with  $\mu(F_x) = x_{i,j}$ ,  $i \neq m$ , all right turns of  $F$  are in  $c(F)$ . The equality can be written more precisely as:

$$C_{i,j}^{k'} = \sum_{k_1+k_2=k'} \binom{m-i}{k_1} \binom{n-j}{k_1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(i, n), (m, j)\})_{k_2} \quad (9)$$

In the case that  $i \neq m$ ,  $\mathcal{P}(\{(1, 2), (1, 1)\}, \{(i, n), (m, j)\})_{k_2}$  can be computed directly using [9, Theorem 2] which says that for  $a < c$  and  $b \geq d$

$$\begin{aligned} & \mathcal{P}(\{(1, 2), (1, 1)\}, \{(a, b), (c, d)\})_s = \\ & = \sum_{s_1+s_2=s} \det \begin{pmatrix} \binom{a-1}{s_1} \binom{b-2}{s_1} & \binom{c-2}{s_1-1} \binom{d-1}{s_1} \\ \binom{a}{s_2+1} \binom{b-2}{s_2} & \binom{c-1}{s_2} \binom{d-1}{s_2} \end{pmatrix} \end{aligned} \quad (10)$$

When  $i = m$ ,  $C_{i,j}^{k'}$  can be computed inductively in the following way:

$$\begin{aligned} C_{m,j}^{k'} & = \\ & = \sum_{s=1}^{n-j-1} \sum_{h=1}^{m-1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-h, n-s), (m, j)\})_{k'-1} \\ & + \sum_{h=2}^{m-1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-h, j), (m, j)\})_{k'-1} \\ & + \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-1, j), (m, j)\})_{k'} \end{aligned} \quad (11)$$

Notice that it is the last summand in the expression above that takes into account the fact that the potential right turn at  $y_{m-1,j+1}$  does not belong to  $c(F)$ . Each of the summands in Equation (11) can be evaluated by Equation (10) above.

Now let's rewrite (8) as

$$h_k = C_{m,n-1}^k + \sum_{(i,j), i \neq m} C_{i,j}^{k-1} + \sum_{j \neq n-1} C_{m,j}^{k-1} \quad (12)$$

Using (9) and (11), we find that

$$\begin{aligned}
 h_k &= \\
 &= \sum_{h=2}^{m-1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-h, n-1), (m, n-1)\})_{k-1} \\
 &+ \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-1, n-1), (m, n-1)\})_k \\
 &+ \sum_{(i,j), i \neq m} \left( \sum_{k_1+k_2=k-1} \binom{m-i}{k_1} \binom{n-j}{k_1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(i, n), (m, j)\})_{k_2} \right) \\
 &+ \sum_{1 < j < n-1} \sum_{s=1}^{n-j-1} \sum_{h=1}^{m-1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-h, n-s), (m, j)\})_{k-2} \\
 &+ \sum_{h=2}^{m-1} \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-h, j), (m, j)\})_{k-2} \\
 &+ \mathcal{P}(\{(1, 2), (1, 1)\}, \{(m-1, j), (m, j)\})_{k-1} \\
 &+ \binom{n-2}{k-1} \binom{m-1}{k-1}
 \end{aligned} \tag{13}$$

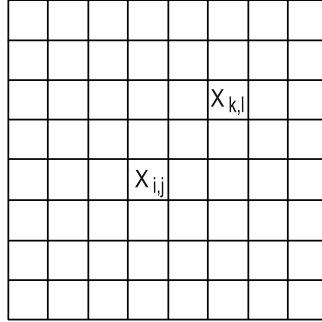
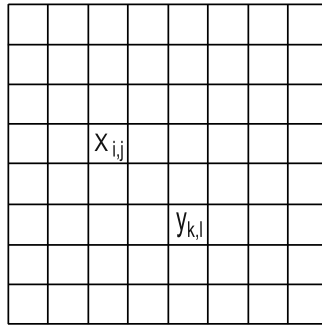
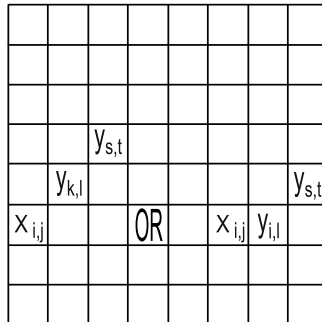
where all  $\mathcal{P}(\{(1, 2), (1, 1)\}, \{(a, b), (c, d)\})_s$  can be computed by Equation (10) above.

Computing, we find the following to be the  $h$ -polynomials for small values of  $m$  and  $n$ :

$m$	$n$	$h(z)$
2	2	$1 + 2z + z^2$
2	3	$1 + 4z + 4z^2$
3	3	$1 + 8z + 18z^2 + 8z^3 + z^4$
3	4	$1 + 12z + 42z^2 + 36z^3 + 9z^4$
4	4	$1 + 18z + 99z^2 + 164z^3 + 99z^5 + 18z^6 + z^7$

It is easy to see by counting directly that when  $m = 2$ , then  $h(z)$  is of the form  $1 + 2(n-1)z + (n-1)^2z^2 = (1 + (n-1)z)^2$ . We conjecture that  $h(z)$  is symmetric in  $z$  precisely when  $m$ . (Since  $I_0$  is prime by [5, Prop. 3.3] and  $R/I_0$  is Cohen-Macaulay, the symmetry of  $h(z)$  is equivalent to  $R/I_0$  being Gorenstein, see [8, Cor. 4.3.8].)

## APPENDIX A. FIGURES

FIGURE 1. Family  $A$ ,  $x_{k,l}$  to NE of  $x_{i,j}$ FIGURE 2. Family  $B$ ,  $y_{k,l}$  to SE of  $x_{i,j}$ FIGURE 3. Family  $C$ ,  $y_{s,t}$  to NE of  $y_{k,l}$  to NE of  $x_{i,j}$ , or,  $y_{s,t}$  to NE of  $y_{i,l}$  to E of  $x_{i,j}$

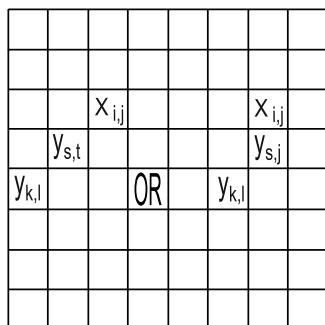


FIGURE 4. Family  $D$ ,  $y_{k,l}$  to SW of  $y_{s,t}$  to SW of  $x_{i,j}$ , or,  $y_{k,l}$  to SW of  $y_{s,j}$  to S of  $x_{i,j}$

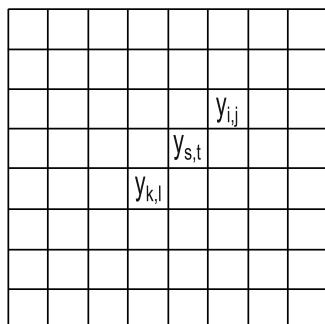


FIGURE 5. Family  $E$ ,  $y_{k,l}$  to SW of  $y_{s,t}$  to SW of  $y_{i,j}$

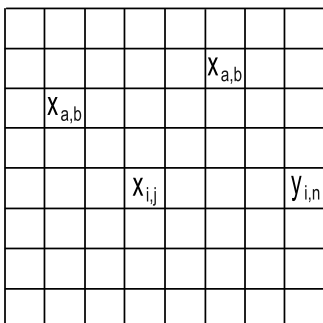


FIGURE 6.

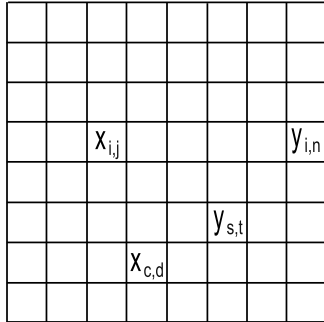


FIGURE 7.

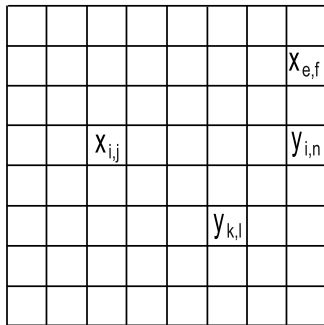


FIGURE 8.

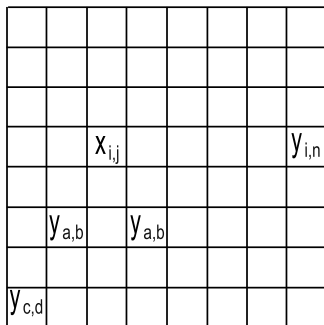


FIGURE 9.



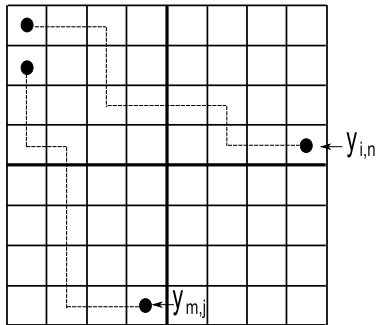


FIGURE 13.

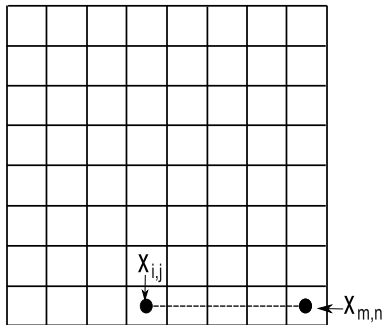


FIGURE 14.

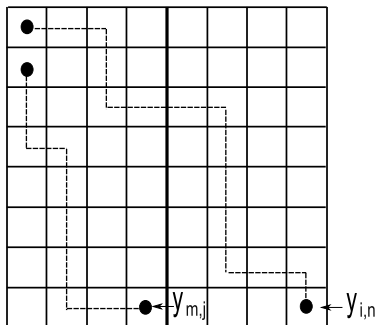


FIGURE 15.

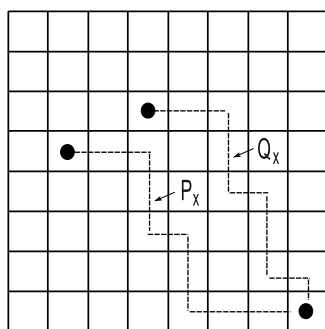


FIGURE 16.

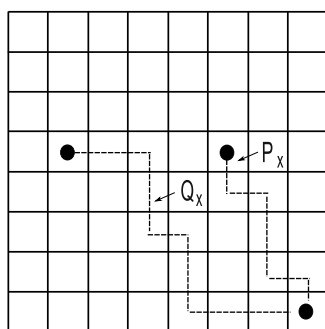


FIGURE 17.

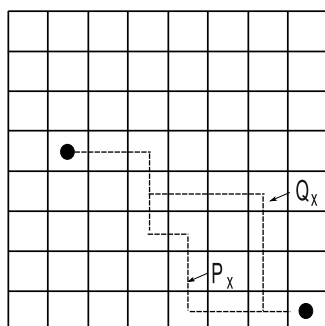


FIGURE 18.

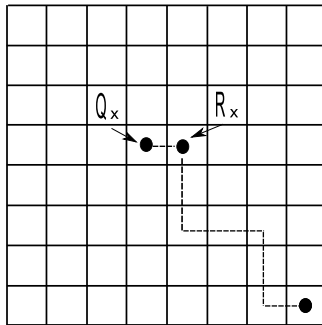


FIGURE 19.

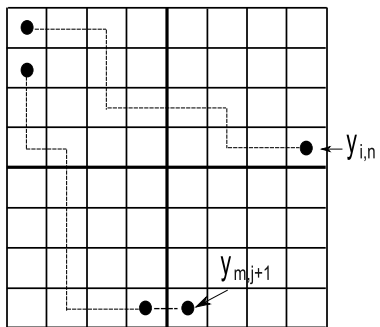


FIGURE 20.

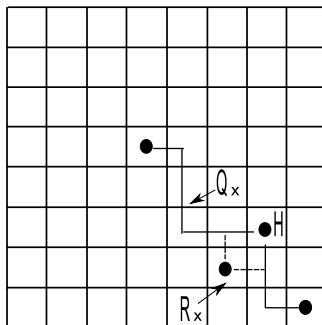


FIGURE 21.



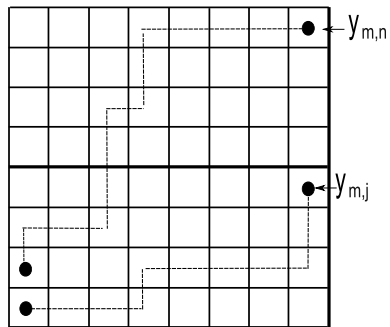


FIGURE 25.

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DEPT. OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY NORTHRIDGE, NORTHRIDGE CA 91330, U.S.A., CURRENT ADDRESS: DEPT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SANTA BARBARA, SANTA BARBARA CA 93106

*E-mail address:* boyan@math.ucsb.edu