THE ALGEBRA GENERATED BY THREE COMMUTING MATRICES

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Abstract. We present a survey of an open problem concerning the dimension of the algebra generated by three commuting matrices.

This article concerns a problem in algebra that is completely elementary to state, yet, has proven tantalizingly difficult and is as yet unsolved. Consider $\mathbb{C}[A, B, C]$, the $\mathbb{C}$-subalgebra of the $n \times n$ matrices $M_n(\mathbb{C})$ generated by three commuting matrices A, B, and C. Thus, $\mathbb{C}[A, B, C]$ consists of all $\mathbb{C}$-linear combinations of "monomials" $A^i B^j C^k$, where i, j, and k range from 0 to infinity. Note that $\mathbb{C}[A, B, C]$ and $M_n(\mathbb{C})$ are naturally vector-spaces over $\mathbb{C}$; moreover, $\mathbb{C}[A, B, C]$ is a subspace of $M_n(\mathbb{C})$. The problem, quite simply, is this: Is the dimension of $\mathbb{C}[A, B, C]$ as a $\mathbb{C}$ vector space bounded above by n?

Note that the dimension of $\mathbb{C}[A, B, C]$ is at most n^2, because the dimension of $M_n(\mathbb{C})$ is n^2. Asking for the dimension of $\mathbb{C}[A, B, C]$ to be bounded above by n when A, B, and C commute is to put considerable restrictions on $\mathbb{C}[A, B, C]$: this is to require that $\mathbb{C}[A, B, C]$ occupy only a small portion of the ambient $M_n(\mathbb{C})$ space in which it sits.

Actually, the dimension of $\mathbb{C}[A, B, C]$ is already bounded above by something slightly smaller than n^2, thanks to a classical theorem of Schur ([16]), who showed that the maximum possible dimension of a commutative $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ is $1 + \lfloor n^2/4 \rfloor$. But n is small relative even to this number.

To understand the interest in n being an upper bound for the dimension of $\mathbb{C}[A, B, C]$, let us look more generally at the dimension of the $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ generated by k-commuting matrices. Let us start with the $k = 1$ case: note that "one commuting matrix" is just an arbitrary matrix A. Recall that the Cayley-Hamilton theorem tells us that A^n is a linear combination of I, A, $\ldots$, A^{n-1}, where I stands for the identity matrix. From this, it follows by repeated reduction that A^{n+1}, A^{n+2}, etc. are all linear combinations of I, A, $\ldots$, A^{n-1} Thus, $\mathbb{C}[A]$, the $\mathbb{C}$-subalgebra of $M_n(\mathbb{C})$ generated by A, is of dimension at most n, and this is just a simple consequence of Cayley-Hamilton.

The case $k = 2$ is therefore the first significant case. It was treated by Gerstenhaber ([4]) as well as Motzkin and Taussky-Todd ([13], who proved independently that the variety of commuting pairs of matrices is irreducible. It follows from this that if A and B are two commuting matrices, then too,

$\mathbb{C}[A, B]$ has dimension bounded above by n. (We will study their sequence of ideas in some depth later in this article.)

Thus, for both $k = 1$ and $k = 2$, our algebra dimension is bounded above by n. Hopes of the dimension of the algebra generated by k commuting matrices being bounded by n for much wider ranges of k were dashed by Gerstenhaber himself: He cited an example of a subalgebra of $M_n(\mathbb{C})$, for $n \geq 4$, generated by $k \geq n$ commuting matrices whose dimension is greater than n. His example easily extends, for each $n \geq 4$ and $k \geq 4$, to a subalgebra of $M_n(\mathbb{C})$ generated by k commuting matrices whose dimension is greater than n, and we give this example here: Write $E_{i,j}$ for the matrix that has zeroes everywhere except for a 1 in the (i, j) slot. (These matrices form a $\mathbb{C}$-basis of $M_n(\mathbb{C})$.) Assume $n \geq 4$, and take $A = E_{1,3}$, $B = E_{1,4}$, $C = E_{2,3}$, and $D = E_{2,4}$. Then A, B, C, and D are linearly independent, and the product of any two of them is zero. In particular, they commute pairwise, and the linear subspace spanned by A, B, C, and D is closed under multiplication. Adding the identity matrix to the mix to get a "1" in our algebra, we find that $\mathbb{C}[A, B, C, D]$ is the $\mathbb{C}$ subspace of $M_n(\mathbb{C})$ with basis I, A, B, C, and D—a five-dimensional algebra. Thus, when $n = 4$, we already have our counterexample for the $k = 4$ case. For larger values of n, this example can be modified by taking A to have 1 in the slot $(1, 3)$ along with nonzero elements $a_5, \ldots, a_n$ in the diagonal slots $(5, 5), \ldots, (n, n)$, chosen so that $a_5^2, \ldots, a_n^2$ are pairwise distinct. The matrices A, B, C, and D will still commute, and it is a short calculation (a Vandermonde matrix will appear!) that $\mathbb{C}[A, B, C, D]$ will have basis I, A, B, C, D along with $A^2, \ldots, A^{n-3}$—an $(n + 1)$-dimensional algebra.

Further, taking $E = F = \cdots = D$ in the example above, we find trivially that for any $k \geq 4$, there exists k commuting matrices which generate a $\mathbb{C}$-algebra of dimension greater than n.

This, then, is the source of our open problem: Yes, the algebra dimension is bounded by n for $k = 1$, and $k = 2$. No, the algebra dimension is not bounded by n for $k \geq 4$. So, what happens for $k = 3$?

Note that without the requirement that A, B, and C commute pairwise, this question would have an immediate answer: already, with $k = 2$, there are easy examples of matrices A and B (that do not commute) for which the algebra they generate is the whole algebra $M_n(\mathbb{C})$, so in particular, of dimension n^2. For instance, take A to be a diagonal matrix with entries that are pairwise distinct, and let B be the permutation matrix corresponding to the cyclic permutation $(1\ 2\ \ldots\ n)$, i.e., the matrix with 1 in the slots $(i, i-1)$ for $i = 2, \ldots, n - 1$ and in the slot $(1, n)$, and zeros everywhere else. One checks (Vandermonde again!) that the matrices $A^i B^j$, for $i, j = 0, \ldots, n-1$ are linearly independent, thus giving an algebra of full dimension n^2.

It is worth noting that matrices of the form $E_{1,3}$, $E_{1,4}$, $E_{2,3}$, $E_{2,4}$ of $M_4(\mathbb{C})$ that arise in the example above quoted by Gerstenhaber play a significant role in the context of commutative subalgebras of $M_n(\mathbb{C})$. More generally, we may partition our $n \times n$ matrix into four blocks of equal (or nearly

equal) sizes and consider the "north-east" block: If $n = 2m$, our north-east block will consist of slots form the first m rows and last m columns. If $n = 2m + 1$, our north-east block will consist of slots from the first m rows and last $m + 1$ columns, or else, from the first $m + 1$ rows and last m columns (we may pick either one). If we consider the $\lfloor n^2/4 \rfloor$ matrices $E_{i,j}$ corresponding to the various slots (i, j) in this block, then it is clear that they are linearly independent and the product of any two of these matrices is zero. These matrices hence commute, and the linear subspace of $M_n(\mathbb{C})$ spanned by them is closed under multiplication. Adding constant multiples of the identity to this space so as to have a "1," we therefore get a commutative subalgebra of $M_n(\mathbb{C})$ of the maximum dimension $1 + \lfloor n^2/4 \rfloor$ possible by Schur's theorem. Schur had also shown that any commutative subalgebra of dimension $1 + \lfloor n^2/4 \rfloor$ must be similar to the algebra generated as above by the matrices $E_{i,j}$ coming from the north-east block.

(In basis-free terms, this corresponds to taking a decomposition of $V \cong \mathbb{C}^n = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of as equal dimensions as possible, and considering all $\{f \in \mathrm{End}_{\mathbb{C}}(V) \mid f(V_2) \subseteq V_1,\ f(V_1) = 0\}$, along with the endomorphisms representing multiplication by constants.)

Jacobson ([10]) later gave an alternative proof of Schur's theorem on the maximum dimension of a commutative subalgebra that is valid for any field F, and showed that if F is not imperfect of characteristic two, then too, any commutative subalgebra F-subalgebra of $M_n(F)$ of the maximum dimension $1 + \lfloor n^2/4 \rfloor$ is conjugate to the algebra generated as above by the matrices $E_{i,j}$ coming from the north-east block.

It is worth remarking in this context that Schur's result was further generalized to the case of artinian rings by Cowsik ([2]): he showed that if A is an artinian ring with a faithful module of length n, then A has length at most $1 + \lfloor n^2/4 \rfloor$. Cowsik was answering a question raised by Gustafson, who had given ([7]) a representation-theoretic proof of Schur's theorem; Gustafson had also proven a related interesting fact: the dimension of a *maximal* commutative subalgebra of $M_n(\mathbb{C})$ is at least $n^{2/3}$.

Other proofs of Schur's theorem have also been given. See [1], [11], [14], or [19], for instance.

An open problem can be interesting (and significant) because it represents a critical gap in a larger conceptual framework that must be filled before the framework can stand: the missing link in a big theory. Alternatively, an open problem could be interesting because its solution has the potential to involve techniques from other areas and to shed light on and raise new questions in other areas. The problem on the bound of the dimension of $\mathbb{C}[A, B, C]$ falls into the second category. Quite specifically, the most significant attacks on this problem have involved the analysis of the algebraic variety of commuting triples of matrices, and interestingly, have spun off investigations into jet schemes of determinantal varieties and of commuting pairs of matrices.

To get a feel for the connection between our open question and matrix varieties (i.e., the solution set in some large dimensional affine space to

polynomial equations defined by matrices–we will see examples below), let us consider the proofs of Taussky-Todd and Motzkin, and of Gerstenhaber that the algebra $\mathbb{C}[A, B]$ generated by two commuting $n \times n$ matrices A and B is of dimension at most n. View pairs of matrices (A, B) as points of affine $2n^2$ dimensional space $\mathbb{C}^{2n^2}$ by viewing the set of entries of A and of B strung together in some fixed order as coordinates of the corresponding point. The set of *commuting* pairs (A, B) correspond to solutions of the n^2 equations arising from the entries of $XY - YX = 0$, where X and Y are generic matrices with entries $x_{i,j}$ and $y_{i,j}$. These equations are polynomial equations in the $x_{i,j}$ and $y_{i,j}$ (in fact, they are bilinear in the $x_{i,j}$ and $y_{i,j}$). Thus, the set of commuting pairs of $n \times n$ matrices (A, B) naturally has the structure of an algebraic variety, which we will denote $C(2, n)$.

Both Taussky-Todd and Motzkin, and Gerstenhaber actually proved that $C(2, n)$ is irreducible. Let us see how their analysis of $C(2, n)$ leads to our desired bound on the dimension of $\mathbb{C}[A, B]$. The proof based on $C(2, n)$ that $\mathbb{C}[A, B]$ has dimension at most n proceeds along the steps below. Both sets of authors use essentially the same set of ideas, with the slight difference that Taussky-Todd and Motzkin use matrices with distinct eigenvalues instead of "1-regular" matrices in steps (1) and (2):

(1) Show first that $\mathbb{C}[A, B]$ has dimension exactly n if A has each eigenvalue of A appears in exactly one Jordan block. (Recall from elementary matrix theory that A has this property precisely when the minimal polynomial of A coincides with the characteristic polynomial of A, i.e., if the algebra $\mathbb{C}[A]$ has dimension exactly n. Such a matrix A is said to be 1-regular.)
(2) Show that the set U of points (A, B) where A is 1-regular is a dense subset (in a suitable topology) of $C(2, n)$. (This is the step that shows the irreducibility of $C(2, n)$; we will consider irreducibility later.)
(3) Show that if $\mathbb{C}[A, B]$ has dimension exactly n, and therefore at most n, for all points (A, B) in a dense subset of $C(2, n)$, then $\mathbb{C}[A, B]$ must have dimension at most n on all of $C(2, n)$.

The topology used is the well known Zariski topology on $\mathbb{C}^{2n^2}$, where a set is closed iff it is the solution set of a system of polynomial equations (in $2n^2$ variables). An open set in this topology is thus the union of sets $D(f)$, where f is a polynomial, and $D(f)$ consists of all points where f is nonzero. To say that the set U in (2) above is dense in $C(2, n)$ in the Zariski topology is therefore to say that if a polynomial vanishes identically on U, then it must vanish identically on $C(2, n)$.

We will describe steps (1), (2), and (3) below and indicate the difficulties in extending these steps to the corresponding variety of commuting triples of matrices.

Step (1). The form of a typical matrix in the centralizer of a given matrix A (when A is described in Jordan form) is well-known and very concrete

(we will not reproduce it here, but see [3] for instance), and it follows from this description that if A is 1-regular, then any matrix B that commutes with A must be a polynomial in A. Described differently, B is already in the algebra $\mathbb{C}[A]$, that is $\mathbb{C}[A,B] = \mathbb{C}[A]$. But $\mathbb{C}[A]$ is of dimension n as A is 1-regular, so $\mathbb{C}[A,B]$ is of dimension n.

Step (2). This is the key step. Once again, one refers to the known form of matrices centralizing a given matrix to observe that given any matrix B, one can find a 1-*regular* matrix A' that commutes with B. (Determining such an A' is actually very easy, although we will not give a recipe for doing this here). So, given an arbitrary point (A,B) in $C(2,n)$, i.e., a commuting pair of matrices (A,B), consider the line L in $\mathbb{C}^{2n^2}$ described by $((1-\lambda)A' + \lambda A, B)$, where λ varies through $\mathbb{C}$ and A' is some 1-regular matrix that commutes with B. Since B and A' commute, the matrices B and $(1-\lambda)A' + \lambda A$ also commute for any λ, i.e, the entire line L lies in $C(2,n)$.

Now consider what it means for a matrix A to be 1-regular. It means that $\mathbb{C}[A]$ must be of dimension n, that is, the matrices $1, A, \ldots, A^{n-1}$ must be linearly independent. In particular, writing each of the matrices 1, A, A^2, ... as an $n^2 \times 1$ (column) vector and assembling all n vectors together, we get an $n^2 \times n$ matrix $M(A)$, and to say that $1, A, \ldots, A^{n-1}$ should be linearly independent is to say that $M(A)$ must have rank n. Thus, $M(A)$ should have the property that at least one of its $n \times n$ minors should be nonzero. Since these minors are polynomials in the entries of $M(A)$, which in turn are polynomials in the entries of A, this translates into an open set condition in the Zariski topology: A, viewed as a point in $\mathbb{C}^{n^2}$, must live in the union of the various open sets in which some $n \times n$ minor of $M(A)$ is nonzero.

Let us apply these ideas to the first coordinates $(1-\lambda)A' + \lambda A$ of the line L above. Let us consider those values of λ for which $(1-\lambda)A' + \lambda A$ is 1-regular. This certainly happens when $\lambda = 0$, by our choice of A'. But more: taking $(1-\lambda)A' + \lambda A$ for A in the paragraph above, the various $n \times n$ minors of $M((1-\lambda)A' + \lambda A)$ are now polynomials in λ. At least one of these polynomials is nonzero, since $\lambda = 0$ is not a solution to at least one of them. But a nonzero polynomial in one variable has only finitely many roots, and hence, all but finitely many λ are nonroots of this polynomial. Put differently, for all but finitely many λ, our matrix $(1-\lambda)A' + \lambda A$ must be 1-regular. Thus, almost all points of L are in U.

Finally, we will show that any point (A,B) in $C(2,n)$ is in the closure of U. Let A' and L be as in the arguments above. Then almost all points of L are in U as we have seen. Let f be any polynomial (in $2n^2$ variables) that is zero on U. Substituting the general point of L into f, we get a new polynomial g in the single variable λ. Since all but finitely many points of L are in U, we find that g is zero for almost all values of λ. Invoking the fact that a nonzero polynomial in a single variable has only finitely many

zeroes, we find $g(\lambda)$ is identically zero. Put differently, f must be zero on the entire line L, and in particular, on the point (A, B) corresponding to $\lambda = 1$. Since (A, B) was arbitrary in $C(2, n)$, we find that any polynomial (in $2n^2$ variables) that vanishes on U must vanish on $C(2, n)$, that is, U is indeed dense in the Zariski topology in $C(2, n)$. (In particular, the closure of U in $C(2, n)$ is all of $C(2, n)$.)

Step (3). We only need to show that the condition that $\mathbb{C}[A, B]$ have dimension at most n is equivalent to a set of polynomial conditions on the corresponding point (A, B) of the variety $C(2, n)$. Then, if these conditions are satisfied on any dense subset S of $C(2, n)$, they must be satisfied on the (Zariski) closure of S, i.e., on all of $C(2, n)$. (In particular, since these polynomial conditions are satisfied on our open set U (by (1)), and since the closure of U in $C(2, n)$ is all of $C(2, n)$ (by (2)), they will be satisfied on all of $C(2, n)$. Thus, the dimension of $\mathbb{C}[A, B]$ will indeed by bounded by n for all commuting pairs (A, B).) To see how the upper bound on the dimension translates to a set of polynomial conditions, we repeat the ideas in step (2) above. Observe that $\mathbb{C}[A, B]$ is spanned by the n^2 products $A^i B^j$ for $0 \leq i, j \leq n - 1$ (note that A and B commute, and by Cayley-Hamilton, powers A^i and B^i for $i \geq n$ can be written as linear combinations of the powers A^i and B^i respectively, for $0 \leq i \leq n - 1$–a fact we have already considered above). As in the proof of (2) above, collect each $A^i B^j$ as an $n^2 \times 1$ (column) vector, and assemble all n^2 such vectors into an $n^2 \times n^2$ matrix $M(A, B)$. Then, the condition that $\mathbb{C}[A, B]$ has dimension at most n translates to $M(A, B)$ having rank at most n, which is now equivalent to all $(n + 1) \times (n + 1)$ minors of $M(A, B)$ *vanishing*. The vanishing of each of these minors is of course a polynomial condition on the entries of A and B. This concludes step (3).

Since the dimension problem for three commuting matrices is still open, these arguments must somehow fail, or at least not extend in any obvious manner, when we consider the corresponding algebraic variety $C(3, n)$ of commuting triples of matrices. What fails? Steps (1) and (3) go through easily: if A is 1-regular and if B and C commute with A, then both B and C are in $\mathbb{C}[A]$, and $\mathbb{C}[A, B, C]$ is hence of dimension at most n; similarly, $\mathbb{C}[A, B, C]$ is spanned by the matrices $A^i B^j C^k$ for $0 \leq i, j, k \leq n - 1$, and collecting each of $A^i B^j C^k$ into an $n^2 \times 1$ vector and assembling all n^3 such into an $n^2 \times n^3$ matrix $M(A, B, C)$, it is clear that the condition that $\mathbb{C}[A, B, C]$ be of dimension at most n translates into the condition that the $(n + 1) \times (n + 1)$ minors of $M(A, B, C)$ vanish, which is a set of polynomial conditions on the entries of A, B, and C. It turns out, however, that step (2) actually *fails* when we consider three commuting matrices! The corresponding set U consisting of triples (A, B, C) where A is 1-regular is no longer dense in $C(3, n)$, at least, for most values of n. This makes the problem hard and interesting!

Here, precisely, is what is known. Let us bring in irreducibility: recall that an algebraic variety X is said to be irreducible if it cannot be written

as $X_1 \cup X_2$ where X_1 and X_2 are themselves algebraic varieties, i.e., solution sets of systems of polynomial equations. If X is not irreducible, we say X is reducible. (Every algebraic variety is a finite union of irreducible varieties, so we may think of irreducible varieties as analagous to prime numbers in the sense of their being building blocks.) Writing $C(k,n)$ for the variety of commuting k-tuples of $n \times n$ matrices for general k, and writing $U(k)$ for the corresponding subset of k-tuples where the first matrix is 1-regular, it turns out that $U(k)$ being dense in $C(k,n)$ is equivalent to $C(k,n)$ being irreducible. (We will not show this equivalence here; as we have already noted, step (2) above effectively proves that $C(2,n)$ is irreducible.) Guralnick ([5]) showed using a very pretty argument that $C(3,n)$ is *reducible* for $n \geq 32$. (Holbrook and Omladič ([9]) later observed that Guralnick's proof really shows that $C(3,n)$ is reducible for $n \geq 29$.) On the other hand, due to the work of several authors ([5], [6], [9], [8], and most recently, Šivic in [18]), it is known that $C(3,n)$ is *irreducible* for all n upto 10. (Thus, the algebra generated by three commuting $n \times n$ matrix for $n \leq 10$ is indeed bounded by n.)

The irreducibility of $C(3,n)$ is thus itself an open problem for $11 \leq n \leq 28$. It would be very useful if the components (the irreducible constituents) of $C(3,n)$ for $n \geq 29$ can be concretely described, for then, one could potentially analyze the dimension of $\mathbb{C}[A,B,C]$ on each component. But such a description seems hopelessly difficult at this point, because the variety $C(3,n)$ has not yielded much structure that might facilitate a concrete listing of its components.

Working in a different direction, Neubauer and this author ([14]) showed that the variety of commuting pairs in the centralizer of a 2-regular matrix is irreducible. (A matrix is r-regular if each eigenvalue appears in at most r blocks.) This variety shows up naturally as a subvariety of $C(3,n)$: it is the variety of all commuting triples (A,B,C) where one of the matrices, say C, has been *fixed* to be a specific 2-regular matrix. The irreducibility of this subvariety then shows (using essentially the same arguments described above behind the proof that the algebra generated by two commuting matrices is at most n-dimensional) that the dimension $\mathbb{C}[A,B,C]$ is indeed bounded by n if one of A, B, or C is 2-regular (and more generally, if any two of A, B, C commute with a 2-regular matrix). It turned out that this particular variety is related to the variety of jets over certain determinantal varieties (determinantal varieties are varieties defined by the vanishing of certain sized minors of a generic $n \times n$ matrix, and jets over such varieties are like algebraic tangent bundles over such varieties). This was very pleasing, and led this author to a broader study of such jet varieties([12]). Meanwhile, Šivic ([18]) showed that the variety of commuting pairs in the centralizer of a 3-regular matrix is also irreducible (which implies a result for the dimension of $\mathbb{C}[A,B,C]$ analagous to the result in the 2-regular case above), but the variety of commuting pairs in the centralizer of an r-regular

matrix is *reducible* for $r \geq 5$. The $r = 4$ case is open, although, there are some partial results in [18].

Working in yet a different direction, Šivic and this author ([17]) considered jet schemes over the commuting pairs variety $C(2, n)$. These varieties also appear naturally as subvarieties of $C(3, n)$, as the set of triples where one of the matrices is a fixed nilpotent matrix whose Jordan blocks all have the same size. They showed that for large enough n, these subvarieties are all reducible, but are indeed irreducible if $n \leq 3$.

To the best of this author's knowledge, this is the current state of the art in the subject. The variety $C(3, n)$ has indeed proved to be a very hard object to tackle, even as it has thrown off interesting subproblems, and in special cases, has exhibited connections to other interesting varieties like jet schemes over determinantal varieties and over the commuting pairs variety. The analysis of $C(3, n)$, as well as the original problem, namely whether $\mathbb{C}[A, B, C]$ has dimension at most n when A, B, and C commute, is in need of fresh ideas and approaches.

References

[1] J. Barria and P. Halmos, Vector bases for two commuting matrices, *Linear and Multilinear Algebra,* **27** 1990, 147–157.

[2] R.C. Cowsik, A short note on the Schur-Jacobson theorem, *Proc. Amer. Math. Soc.,* **118** (1993), 675–676.

[3] F.R. Gantmacher, The Theory of Matrices, Vol. 1, Chelsea Publishing Company, New York, 1959.

[4] M. Gerstenhaber, On dominance and varieties of commuting matrices, *Annals of Mathematics,* **73** (1961), 324–348.

[5] R.Guralnick, A note on commuting pairs of matrices, *Linear and Multilinear Algebra,* **31** (1992), 71–75.

[6] R. M. Guralnick and B. A. Sethuraman, Commuting pairs and triples of matrices and related varieties, *Linear algebra and its applications,* **310** (2000), 139–148.

[7] W.H. Gustafson, On maximal commutative algebras of linear transformations, *Journal of Algebra,* **42** (1976), 557–653.

[8] Y. Han, Commuting triples of matrices, *Electronic journal of linear algebra,* **13** (2005), 274–343.

[9] J. Holbrook and M. Omladič, Approximating commuting operators, *Linear algebra and its applications,* **327** (2001), 131–149.

[10] N. Jacobson, Schur's theorems on commuting matrices, *Bull. Amer. Math. Soc.,* **50** (1944), 431–436.

[11] T. Laffey and S. Lazarus, Two-generated commutative matrix subalgebras, *Linear Algebra and Applications,* **147** (1991), 249–273.

[12] Tomaž Košir and B.A. Sethuraman, Determinantal varieties over truncated polynomial rings, Journal of Pure and Applied Algebra, **195** (2005), 75–95.

[13] T. Motzkin and O. Taussky-Todd, Pairs of matrices with property L. II, *Transactions of the AMS,* **80** (1955), 387–401.

[14] Michael Neubauer and David Saltman, Two-generated commutative subalgebras of $M_n(F)$, *Journal of Algebra,* **164** (1994), 545–562.

[15] M.J. Neubauer and B.A. Sethuraman, Commuting pairs in the centralizers of 2-regular matrices, *Journal of Algebra,* **214** (1999), 174–181.

[16] I. Schur, Zur Theorie der vertauschbaren Matrizen, *J. Reine Angew. Math.,* **130** (1905), 66-76.

[17] B.A. Sethuraman and Klemen Šivic, Jet schemes of the commuting matrix pairs scheme, *Proc. Amer. Math. Soc.* **137** (2009), 3953–3967.

[18] Klemen Šivic, Varieties of triples of commuting matrices, Ph.D. Thesis, University of Ljubljana, Slovenia, (2001).

[19] A.R. Wadsworth, On commuting pairs of matrices, *Linear and Multilinear Algebra,* **27** (1990), 159–162.

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