COMMUTING PAIRS IN THE CENTRALIZERS OF 2-REGULAR MATRICES

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Abstract. In $M_n(k)$, k an algebraically closed field, we call a matrix l-regular if each eigenspace is at most l-dimensional. We prove that the variety of commuting pairs in the centralizer of a 2-regular matrix is the direct product of various affine spaces and various determinantal varieties $\mathcal{Z}_{l,m}$ obtained from matrices over truncated polynomial rings. We prove that these varieties $\mathcal{Z}_{l,m}$ are irreducible, and apply this to the case of the k-algebra generated by three commuting matrices: we show that if one of the three matrices is 2-regular, then the algebra has dimension at most n. We also show that such an algebra is always contained in a commutative subalgebra of $M_n(k)$ of dimension exactly n.

1. Introduction.

Recall that if k is a field, a matrix $A \in M_n(k)$ is said to be *regular* (or *nonderogatory*) if the minimal polynomial is equal to its characteristic polynomial, or equivalently, if $V \cong k^n$ is a cyclic $k[A]$ module, or equivalently, if each eigenspace of A is one-dimensional. Generalizing this, we will call a matrix A *l-regular* if each eigenspace of A is at most l-dimensional, or equivalently, if $V \cong k^n$ is generated by at most l elements as a $k[A]$ module. (Thus, a regular matrix is now a 1-regular matrix in this terminology.)

We study, in this paper, the variety of commuting pairs in the centralizer of A, for A a 2-regular matrix. We show that this variety is closely related to determinantal varieties, and we prove that it is irreducible. Our goal will be the following question first raised by Gerstenhaber ([3]): if A, B, and C are three commuting matrices in $M_n(k)$, is $\dim_k k[A, B, C]$ bounded by n? We show that the answer is yes if one of the matrices is 2-regular (and more generally, if some two of the matrices commutes with a 2-regular matrix). Moreover, we show that such an algebra is always contained in a commutative subalgebra of dimension exactly n, from which it follows, of course, that the centralizer of such an algebra has dimension at least n.

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Interestingly, we find that in contrast to the 2-generated case, it is possible for $k[A, B, C]$ to be of dimension n but Cent $(k[A, B, C])$ to be of dimension greater than n.

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2. Preliminaries on 2-regular matrices.

We show that the set of 2-regular (more generally l-regular) matrices form an open set in $M_n(k)$ (viewed as $\mathbf{A}^{n^2}$), and we review the description of the centralizer of a 2-regular matrix.

Proposition 1. *For any $l \geq 1$, the set of l-regular matrices in $M_n(k)$ (viewed as $\mathbf{A}^{n^2}$) is Zariski-open.*

Proof. We give an improvement over our own proof that is due to Guralnick: Let $V \cong k^n$. For each l-tuple $p = (v_1, \ldots, v_l)$ in V^k, let $U(p)$ be the set of those matrices $A \in M_n(k)$ for which $v_1, \ldots, v_l$ generate V as a k[A] module. $U(p)$ is an open set, since it consists of those matrices for which the $n \times nl$ matrix with columns $A^i v_j$ ($i = 0, \ldots, n-1$, $j = 1, \ldots, l$) has rank exactly n. The set of l-regular matrices is simply the union over all p of the open sets $U(p)$. ☐

Notation 2. Given a matrix C, we will let $\mathcal{C}^2_C$ denote the variety of commuting pairs in the centralizer of C.

We recall the following elementary facts, which we state without proof:

Lemma 3. *Given $C \in M_n(k)$, let $\lambda_1, \ldots, \lambda_s$ be its distinct eigenvalues. Let $V \cong k^n$ decompose into $V_1 \oplus \cdots \oplus V_s$ as $k[C]$ modules, where the V_i are the elements annihilated by a suitable power of $C - \lambda_i I$. Write C_i for $C|_{V_i}$. Then*

1. $End_{K[C]} V \cong End_{K[C_1]} V_1 \times \cdots \times End_{K[C_s]} V_s$.
2. $\mathcal{C}^2_C \cong \mathcal{C}^2_{C_1} \times \cdots \times \mathcal{C}^2_{C_s}$.
3. *For any λ_i for which C_i is 1-regular, $End_{k[C_i]} V_i \cong k[C_i]$, an algebra of dimension $\dim_k V_i$ (= t, say), and $\mathcal{C}^2_{C_i} \cong \mathbf{A}^t \times \mathbf{A}^t$.*

The lemma above allows us to reduce our study of commuting pairs in the centralizer of 2-regular matrices to the following situation: C is nilpotent, with a 2-dimensional eigenspace. Thus, V decomposes as a $k[C]$ module as $V_1 \oplus V_2$, and we assume that $\dim_k V_1 = p \geq \dim_k V_2 = m$.

Writing R for $K[C]$, we have $R \cong k[z]/z^p$, and $V_1 \cong R$ and $V_2 \cong R/z^m R$ as R-modules. The following result is classical (see [2, Chapter 8, §2] for instance):

Lemma 4. *Every element of $End_R V$ is uniquely described by a matrix*

$$\begin{pmatrix} f_{1,1}(z) & z^e f_{1,2}(z) \\ f_{2,1}(z) & f_{2,2}(z) \end{pmatrix},$$

where $f_{1,1} \in R$, $f_{1,2}$ is in the subspace of R consisting of polynomials of degree less than m, $f_{2,1}$, and $f_{2,2}$ are in $R/z^m R$, and $e = p - m$.

Proof. (*Sketch.*) Let u and w be generators for V_1 and V_2 respectively. For any $A \in \mathrm{End}_R V$, write $Au = f_{1,1}(z)u + f_{2,1}(z)v$ and $Aw = f_{1,2}(z)u + f_{2,2}(z)v$, where $f_{1,1}$ and $f_{1,2}$ are uniquely determined in R and $f_{2,1}$ and $f_{2,2}$ are uniquely determined in $R/z^m R$. Since z^m annihilates w, $f_{1,2}$ is constrained to be in $z^e R$, from which the description of A follows. □

Note that the multiplication in $\mathrm{End}_R V$ corresponds to an obvious multiplication of the matrices described in the lemma. Also, note that when $p = m$, $\mathrm{End}_R V \cong M_2(k[z]/z^m) \cong (M_2(k))[z]/z^m$.

3. Reduction to the 2-regular homogenous case.

We continue in the situation of Lemma 4 above, and show that we may further reduce our study to the homogenous case. (Recall that a matrix A is called *homogenous* if it has only one eigenvalue, and all its Jordan blocks have the same size.)

In the lemma below, we will abuse the notation of Lemma 4 slightly and rewrite the $(1,1)$ slot as $f_{1,1}(z) + z^m f'_{1,1}(z)$, where $f_{1,1}$ is simply the first m monomials of the $(1,1)$ slot.

Lemma 5. *Given*

$$A = \begin{pmatrix} f_{1,1}(z) + z^m f'_{1,1}(z) & z^e f_{1,2}(z) \\ f_{2,1}(z) & f_{2,2}(z) \end{pmatrix},$$

and

$$B = \begin{pmatrix} g_{1,1}(z) + z^m g'_{1,1}(z) & z^e g_{1,2}(z) \\ g_{2,1}(z) & g_{2,2}(z) \end{pmatrix}$$

in $End_R V$, the matrices A and B commute if and only if the following matrices

$$\overline{A} = \begin{pmatrix} f_{1,1}(z) & f_{1,2}(z) \\ f_{2,1}(z) & f_{2,2}(z) \end{pmatrix}$$

and

$$\overline{B} = \begin{pmatrix} g_{1,1}(z) & g_{1,2}(z) \\ g_{2,1}(z) & g_{2,2}(z) \end{pmatrix}$$

commute in $M_2(k[z]/z^m)$.

Proof. This is an easy computation. □

Corollary 6. *The variety of commuting pairs in* $End_{k[C]}V$ *is just the product of* $\mathbf{A}^e \times \mathbf{A}^e$ *and the variety of commuting pairs in* $End_{k[\overline{C}]}\overline{V}$*, where* $e = p - m$ *and* $\overline{C}$ *is nilpotent, homogenous, and* 2*-regular (but not* 1*-regular), and* $\overline{V}$ *is a* $2m$*-dimensional space isomorphic to* $k[\overline{C}] \oplus k[\overline{C}]$ *as* $k[\overline{C}]$ *modules.*

Proof. This is clear. □

4. The commuting pairs variety in the homogenous case.

We now assume that C is a nilpotent homogenous 2-regular (but not 1-regular) matrix, and describe the equations of the variety of commuting pairs in the centralizer of C. We observe that this variety is just a determinantal variety over $k[z]/z^m$, and prove that it is irreducible.

We write $V \cong k[C] \oplus k[C]$ as $k[C]$ modules, with $k[C] \cong k[z]/z^m$. Letting $n = \dim_k V$, we find $n = 2m$. The centralizer of C, by our earlier discussions, is isomorphic to $M_2(k[z]/z^m) \cong (M_2(k))[z]/z^m$.

Theorem 7. *Let* C *be as above. Then* $\mathcal{C}_C^2 \cong \mathcal{Z}_{3,m} \times \mathbf{A}^m \times \mathbf{A}^m$*, where* $\mathcal{Z}_{l,m}$ *is the subvariety of* $\mathbf{A}^{2lm}$ *consisting of all* l*-tuples* $\mathbf{p}_i$, $\mathbf{q}_i$ $(i = 0, \ldots, m-1)$*, subject to the following equations in* $k^l \wedge k^l$*:*

$$\sum_{i=0}^{s} \mathbf{p}_i \wedge \mathbf{q}_{s-i} = \mathbf{0} \ (s = 0, \ldots, m-1).$$

The correspondence associates to a commuting pair (A, B)*, with*

$$A = \begin{pmatrix} f_{1,1}(z) & f_{1,2}(z) \\ f_{2,1}(z) & f_{2,2}(z) \end{pmatrix}$$

and

$$B = \begin{pmatrix} g_{1,1}(z) & g_{1,2}(z) \\ g_{2,1}(z) & g_{2,2}(z) \end{pmatrix},$$

the triples

$$\mathbf{p}_i = \begin{pmatrix} f_{1,2,i} \\ f_{2,1.i} \\ f_{1,1,i} - f_{2,2,i} \end{pmatrix} \text{ and } \mathbf{q}_i = \begin{pmatrix} g_{1,2,i} \\ g_{2,1.i} \\ g_{1,1,i} - g_{2,2,i} \end{pmatrix}$$

in $\mathcal{Z}_{3,m}$ *and the point* $(f_{1,1,0}, \ldots, f_{1,1,m-1}, g_{1,1,0}, \ldots, g_{1,1,m-1})$ *in* $\mathbf{A}^m \times \mathbf{A}^m$*. (Here,* $f_{i,j,k}$ *stands for the coefficient of* z^k *in* $f_{i,j}(z)$*.)*

Proof. This is a simple calculation. The commuting relation $AB = BA$ gives us the following equations in $k[z]/z^m$:

$$\begin{aligned} f_{1,2}g_{2,1} &= g_{1,2}f_{2,1} \\ f_{1,1}g_{1,2} + f_{1,2}g_{2,2} &= g_{1,1}f_{1,2} + g_{1,2}f_{2,2} \\ f_{2,1}g_{1,1} + f_{2,2}g_{2,1} &= g_{2,1}f_{1,1} + g_{2,2}f_{2,1} \end{aligned}$$

Expanding in powers of z and collecting like terms, we have the result. □

Remark 8. The varieties $\mathcal{Z}_{l,m}$ may be interpreted as determinantal varieties as follows: We may identify $\mathbf{A}^{2lm}$ with $M_{l,2}(k[z]/z^m)$ by associating to the l-tuples $\mathbf{p}_i$, $\mathbf{q}_i$ ($i = 0, \dots, m-1$) the matrix with columns $\sum_{i=0}^{m-1} \mathbf{p}_i z^i$ and $\sum_{i=0}^{m-1} \mathbf{q}_i z^i$. Then, the defining equations for $\mathcal{Z}_{l,m}$ are precisely the conditions that the exterior product of this matrix vanish.

Our next result is key to our understanding of $\dim_k k[A, B, C]$.

Theorem 9. *The varieties $\mathcal{Z}_{l,m}$ defined in Theorem 7 above are irreducible, of dimension $m(l+1)$.*

Proof. It is easy to see that in the open set U where $\mathbf{p}_0 \neq \mathbf{0}$, the solution to the equations for $\mathcal{Z}_{l,m}$ is given by $\mathbf{q}_s = \sum_{i=0}^{s} \lambda_{s-i}\mathbf{p}_i$ ($s = 0, \dots, m-1$), where the λ_i are arbitrary elements of k. Moreover, $\sum_{i=0}^{s} \lambda_{s-i}\mathbf{p}_i = \sum_{i=0}^{s} \lambda'_{s-i}\mathbf{p}_i$ ($s = 0, \dots, m-1$) if and only if $\lambda_i = \lambda'_i$ ($i = 0, \dots, m-1$). So, U is isomorphic to an open subset of $\mathbf{A}^{lm} \times \mathbf{A}^m$, and is thus irreducible of dimension $m(l+1)$.

When $m = 1$, $\mathcal{Z}_{l,1}$ is just the variety of $l \times 2$ matrices of rank at most 1, a variety that is well known to be irreducible. Now assume that we have shown that $\mathcal{Z}_{l,m-1}$ is irreducible. To show that $\mathcal{Z}_{l,m}$ is irreducible, it suffices to show that the set U defined above is dense in $\mathcal{Z}_{l,m}$, so it suffices to show that any point with $\mathbf{p}_0 = \mathbf{0}$ is in the closure of U. But the equations of $\mathcal{Z}_{l,m}$ show that the subvariety $\mathbf{p}_0 = \mathbf{0}$ is just $\mathcal{Z}_{l,m-1} \times \mathbf{A}^l$, which is irreducible by induction. In particular, the set where $\mathbf{p}_0 = \mathbf{0}$ *and* $\mathbf{q}_0 \neq \mathbf{0}$ is dense in this subvariety. It thus suffices to show that any point with $\mathbf{p}_0 = \mathbf{0}$ and $\mathbf{q}_0 \neq \mathbf{0}$ is in the closure of U. Given such a point $P = (\mathbf{0}, \mathbf{p}_1, \dots, \mathbf{p}_{m-1}, \mathbf{q}_0, \dots, \mathbf{q}_{m-1})$ with $\mathbf{q}_0 \neq \mathbf{0}$, consider the line $Q(s) = (s\mathbf{q}_0, \mathbf{p}_1 + s\mathbf{q}_1, \dots, \mathbf{p}_{m-1} + s\mathbf{q}_{m-1}, \mathbf{q}_0, \dots, \mathbf{q}_{m-1})$. Using the fact that P is in $\mathcal{Z}_{l,m}$, it is easy to see that $Q(s)$ is in $\mathcal{Z}_{l,m}$, and of course, for $s \neq 0$, $Q(s) \in U$. It follows that the closure of U contains $Q(0) = P$. □

Corollary 10. *For C any two regular matrix, $\mathcal{C}_C^2$ is irreducible.*

Proof. This just follows from Lemma 3, Corollary 6, and Theorems 7 and 9 above. □

5. The algebra generated by 3 commuting matrices.

We will study in this section, the algebra generated by a commuting triple A, B, C, where some two of these matrices commute with a 2-regular matrix. Our approach will be via certain subsets of $\mathcal{C}_n^3$, the variety of commuting triples of $n \times n$ matrices. (We note that for $n \geq 32$, $\mathcal{C}_n^3$ is known to be reducible, while for $n \leq 4$, it is known to be irreducible; see [4] and [5].)

For any i, $1 \leq i < n$, let V_i be the open subset of $M_n(k)$ of i-regular matrices. Let π_j $(j = 1, 2, 3)$ represent the projection from $\mathcal{C}_n^3$ to the j-th component, and let $X_i = \pi_1^{-1} V_i$, $Y_i = \pi_2^{-1} V_i$, and $Z_i = \pi_3^{-1} V_i$. Let $U_i = X_i \cap Y_i \cap Z_i$.

Lemma 11. $\overline{U_i} = \overline{X_i} = \overline{Y_i} = \overline{Z_i}$, *where the bars denote the Zariski closures of the respective sets.*

Proof. We will prove that $\overline{U_i} = \overline{X_i}$, the other proofs are similar. It is sufficient to prove that $X_i \subset \overline{U_i}$. Given a triple (A, B, C) with A i-regular, the line $L = \{(A, (1-t)B + tA, (1-t)C + tA)\}$ is contained in $\mathcal{C}_n^3$ and intersects U_i at least in the point $t = 1$. Since U_i is open in $\mathcal{C}_n^3$ and $U_i \cap L$ is nonempty, U_i is dense in L. It follows that $L \subseteq \overline{U_i}$. In particular, $(A, B, C) \in \overline{U_i}$. □

Remark 12. It is easy to see that $\overline{U_{n-1}} = \mathcal{C}_n^3$.

Remark 13. $\overline{U_1}$ is irreducible of dimension $n^2 + 2n$. This follows from the fact that X_1 is irreducible of dimension $n^2 + 2n$, which in turn follows from the fact that any commuting triple (A, B, C), with A a 1-regular matrix, must satisfy $B = f(A)$ and $C = g(A)$ for uniquely determined polynomials f and g of degree at most $n - 1$. (See [4] for instance.)

We have the filtration $\overline{U_1} \subseteq \overline{U_2} \subseteq \cdots \subseteq \overline{U_{n-1}} = \mathcal{C}_n^3$. The following is a consequence of the irreducibility of $\mathcal{C}_C^2$ considered in the previous section.

Proposition 14. $\overline{U_2} = \overline{U_1}$, *so that* $\overline{U_2}$ *is also irreducible of dimension* $n^2 + 2n$.

Proof. Since $U_1 \subset U_2$, it is sufficient to prove that $\overline{U_2} \subset \overline{U_1}$. By Lemma 11 above, it is sufficient to prove that $X_2 \subseteq \overline{U_1}$. Given a commuting triple (A, B, C), with A a 2-regular matrix, note that there exists a point $(D, 0)$ in $\mathcal{C}_A^2$ with D a 1-regular matrix. Thus, $Y_1 \cap \mathcal{C}_A^2$

is nonempty (here, we are viewing $\mathcal{C}_A^2$ as sitting naturally inside $\mathcal{C}_n^3$). Since $\mathcal{C}_A^2$ is irreducible, $\overline{Y_1 \cap \mathcal{C}_A^2}$ is all of $\mathcal{C}_A^2$. But $\overline{Y_1 \cap \mathcal{C}_A^2} \subset \overline{Y_1} = \overline{U_1}$. Thus, $(A, B, C) \in \overline{U_1}$. □

Theorem 15. *Let T be the subset of $\mathcal{C}_n^3$ consisting of triples (A, B, C) in which some two of the three matrices commute with a 2-regular matrix. For $(A, B, C) \in T$, let $\mathcal{A}$ denote the algebra $k[A, B, C]$. We have the following:*

1. $T \subseteq \overline{U_1}$.
2. *$\mathcal{A}$ is contained in a commutative subalgebra of $M_n(k)$ of dimension exactly n.*
3. *The dimension of $\mathcal{A}$ is at most n.*
4. *The dimension of the centralizer of $\mathcal{A}$ is at least n.*

Proof. Suppose A and B commute with a 2-regular matrix D. The line $L = \{(A, B, (1-t)C + tD)\}$ is contained in $\mathcal{C}_n^3$ and intersects Z_2 at least in the point $t = 1$. Since Z_2 is open in $\mathcal{C}_n^3$ and $Z_2 \cap L$ is nonempty, Z_2 is dense in L. It follows that $L \subseteq \overline{Z_2} = \overline{U_2} = \overline{U_1}$, the last two equalities arising from Lemma 11 and Proposition 14. Hence $(A, B, C) \in \overline{U_1}$.

The conditions in 2, 3, and 4 are all polynomial conditions in the coordinates of the matrices A, B, and C. (The proofs that the conditions in 3 and 4 are polynomial conditions is standard, see [8] for instance. For the condition in 2, see [5].) Since these polynomial conditions hold trivially on X_1, they hold on the closure $\overline{X_1} = \overline{U_1}$. Since $T \subset \overline{U_1}$, the statements immediately follow. (Of course, statements 3 and 4 are also direct consequences of 2.) □

6. An example

We show that unlike the two-generated case (see [8, Theorem 1.1]), it is possible for $k[A, B, C]$ to be of dimension n but Cent $(k[A, B, C])$ to be of dimension greater than n. Let C be a nonzero nilpotent homogenous 2-regular (but not 1-regular) matrix satifying $C^2 = 0$. Then $k[C] \cong k[z]/z^2$ and Cent $(C) \cong (M_2(k))[z]/z^2$. Let X_1, X_2, and X_3 be any three matrices in $M_2(k)$ that, together with the identity matrix I_2, form a basis for $M_2(k)$. Take $A = zX_1$ and $B = zX_2$. The linear independence condition shows that $k[A, B, C]$ is spanned by I_2, $z = zI_2$, zX_1, and zX_2. On the other hand, zX_3 centralizes all four of these matrices, so $\dim_k$ Cent $(k[A, B, C]) \geq 5$. (In fact, it easy to check that $\dim_k$ Cent $(k[A, B, C])$ is exactly 5.)

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