Essential Skills Chapter 3

1. **Simplifying the difference quotient** \( \frac{f(x+h) - f(x)}{h} \)  
   Section 3.1
   Example: For \( f(x) = 3 - 4x - 4x^2 \), find \( \frac{f(x+h) - f(x)}{h} \) and simplify completely.
   Answer: \(-4 - 8x - 4h\)

2. **Finding the domain of a function**  Section 3.1
   Example: Find the domain of \( f(x) = \frac{4}{\sqrt{x - 9}} \).
   Answer: \((9, \infty)\)

3. **Using functions as models to make predictions and draw conclusions**  Section 3.1
   Example: If a rock falls from a height of 20 meters on the planet Jupiter, its height \( H \) (in meters) after \( x \) seconds is approximately \( H(x) = 20 - 13x^2 \).
   a. What is the height of the rock after 1 second? Answer: 7 meters
   b. When is the height of the rock 10 meters? Answer: After approximately 0.88 seconds
   c. When does the rock strike the ground? Answer: After approximately 1.24 seconds

4. **Finding information from the graph of a function** Sections 3.2 and 3.3
   Example: Use the graph of the function \( f \) below to answer the following:

   ![Graph of a function](image)

   a. What is \( f(3) \)? Answer: \(-2\)
   b. What is \( f(2) \) approximately? Answer: \( \frac{1}{2} \)
5. **Identifying the relationship between a function and its graph** Section 3.2

**Example:** Let \( f(x) = \frac{x^2 + 2}{x + 4} \)

a. Is the point \((1, 3)\) on the graph of \( f \)? Answer: yes
b. If \( x = 0 \), what is \( f(x) \)? What corresponding point is on the graph of \( f \)? Answers: \( \frac{1}{4}, \ (0, \frac{1}{4}) \)
c. If \( f(x) = \frac{1}{4} \), what corresponding points are on the graph of \( f \)?
d. Find the \( x \) and \( y \) intercepts of the graph of \( f \). Answers: \( x \)-intercept: none, \( y \)-intercept: \( (0, \frac{1}{4}) \)

6. **Finding the average rate of change of a function** Section 3.3

**Example:** Find the average rate of change of the function \( f(x) = \sqrt{1-x} \) on the interval \([-7, 9]\).

Answer: \( -\frac{1}{4} \)

7. **Sketching graphs of basic functions** Section 3.4

**Example:** Sketch the graph of \( f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } x > 1 \end{cases} \).

Answer:

8. **Sketching graphs of basic functions using transformations** Section 3.5

**Example:** Sketch the graph of \( f(x) = (x + 2)^3 - 3 \)
9. **Constructing functions for modeling**  Section 3.6

Example:  A rectangle with width $x$ has a perimeter of 24 inches.

a. Express the length of the rectangle $l$ as a function of $x$.  Answer:  $l(x) = 12 - x$

b. Find the domain of $l$.  Express your answer in interval notation.  Answer:  $(0, 12)$

c. Express the area of the rectangle $A$ as a function of $x$.  Answer:  $A(x) = 12x - x^2$
1. **Identifying properties of linear functions**  Section 4.1

   Example: If \( f(x) = -\frac{3}{2}x + 2 \),
   
   a. Determine the slope and \( y \)-intercept of \( f \).
   b. Use the slope and \( y \)-intercept to graph \( f \).
   c. Determine the average rate of change of \( f \) on the interval \( [3, \frac{4}{3}] \).
   d. Determine whether \( f \) is increasing, decreasing, or constant.

   Answers: a. slope = \( -\frac{3}{2} \), \( y \)-intercept = 2  
              b. start at (0, 2), then go down three and right two  
              to (2, -1)  
              c. \( -\frac{3}{2} \)  
              d. decreasing

2. **Using linear functions as models**  Section 4.1

   Example: In 2002, major league baseball signed a labor agreement with the players. In this agreement, any team whose payroll exceeds $128 million starting in 2005 will have to pay a luxury tax of 22.5\% (for first-time offenses). The linear function \( T(p) = 0.225(p - 128) \) describes the luxury tax \( T \) of a team whose payroll is \( p \) (in millions of dollars).

   a. What is the implied domain of this function?
   b. What is the luxury tax for a team whose payroll is $160 million?
   c. What is the payroll of a team that pays a luxury tax of $11.7 million?

   Answers: a. \( \{ p \mid p \geq 128 \} \)  
              b. $7.2 million  
              c. $180 million

3. **Identifying properties of quadratic functions**  Section 4.3

   Example: Let \( f(x) = -2x^2 - 4x - 3 \).

   a. Express \( f \) in the form \( f(x) = a(x - h)^2 + k \)
   b. Find the vertex of the graph of \( f \).
   c. Find the axis of symmetry of the graph of \( f \).
   d. Find the \( x \) and \( y \) intercepts of the graph of \( f \).
   e. Where is \( f \) increasing and decreasing?
   f. Find the domain and range of \( f \).
   g. Does \( f \) have a maximum or minimum value? What is the maximum/minimum value?
   h. Sketch the graph of \( f \).
Answers:  

a. \( f(x) = -2(x+1)^2 - 1 \)  
b. \((-1, -1)\)  
c. \( x = -1 \)  
d. x-intercepts: none, y-intercept: \((0, -3)\)  
e. increasing on \((-\infty, -1]\), decreasing on \([-1, \infty)\)  
f. domain: \(( -\infty, \infty )\), range: \(( -\infty, -1]\)  
g. maximum; \(-1\)  
h.  

2. 4. **Finding optimal values of quadratic models**  Section 4.3  
**Example:** Paradise Travel Agency’s monthly profit \( P \) (in thousands of dollars) depends on the amount of money \( x \) (in thousands of dollars) spent on advertising per month according to the rule \( P(x) = 7 - 2x(x-4) \). What is Paradise’s maximum monthly profit?  
Answer: \$15,000  

5. **Constructing and using quadratic models**  Section 4.4  
**Example:** A farmer with 2000 meters of fencing wants to enclose a rectangular plot that borders on a straight highway. If the farmer does not fence the side along the highway,  
a. Express the area, \( A \), of the plot as a function of \( x \) (where \( x \) is the side of the plot perpendicular to the highway).  
b. What is the largest area that can be enclosed?  

Answers:  

a. \( A(x) = 2000x - 2x^2 \)  
b. 500,000 square meters
Essential Skills Chapter 5

1. **Graphing polynomial functions**  Section 5.1
   
   **Example:** Let \( f(x) = (x - 2)^2 (x - 3)(x + 1) \).

   a. Find the \( x \) and \( y \) intercepts of the graph of \( f \).
   b. Determine the end behavior of \( f \).
   c. Sketch the graph of \( f \).

   **Answers:**
   
   a. \( x \)-intercepts: \( (2,0), (3,0), (-1,0) \), \( y \)-intercept: \( (0,-12) \)
   
   b. as \( x \to \infty, y \to \infty \); as \( x \to -\infty, y \to \infty \)

   c.

2. **Identifying properties of polynomial functions**  Section 5.1
   
   **Example:** Use the graph of a polynomial function \( P(x) \) to answer the following questions.

   **Diagram:**

   a. What is the minimum degree of \( P(x) \)?
   b. What is the sign of the leading coefficient of \( P(x) \)?
   c. Is the degree of \( P(x) \) odd or even?
   d. Which of the following is most likely to be the formula for \( P(x) \)? (circle one)

   i) \( P(x) = x(x + 2)(x - 2)(x - 3) \)
   
   ii) \( P(x) = x(x + 2)^2(x - 2)(x - 3) \)
   
   iii) \( P(x) = x(x + 2)^2(x - 2)(x - 3)^2 \)
   
   iv) \( P(x) = x(x + 2)(x - 2)^2(x + 3)^2 \)

   **Answers:**
   
   a. 6  b. positive  c. even  d. iii
3. **Graphing rational functions** Sections 5.2 and 5.3

Example: Let $R(x) = \frac{x^2 - x - 12}{x + 1}$.

a. Find the $x$ and $y$ intercepts of the graph of $R$.
b. Find the equations of all asymptotes.
c. Sketch the graph of $R$.

Answers: a. $x$-intercepts: $(4, 0), (-3, 0)$, $y$-intercept: $(0, -12)$
b. vertical asymptote: $x = -1$, horizontal asymptote: none, oblique asymptote: $y = x - 2$

c.

4. **Solving polynomial and rational inequalities** Section 5.4

Example: Solve. $\frac{x}{x + 2} \leq \frac{1}{x}$

Answer: $x \in (-2, -1] \cup (0, 2]$ 

5. **Finding zeros of polynomials** Sections 5.5 and 5.6

Example: Find all the zeros of $P(x) = 2x^3 - 5x^2 + 6x - 2$

Answer: $\frac{1}{2}, 1 + i, 1 - i$

6. **Writing the equation of a polynomial using its zeros** Section 5.6

Example: Write the equation of the fourth degree polynomial $P(x)$ having $1 - i$ a zero, $-2$ a zero of multiplicity 2, and a leading coefficient of 3.

Answer: $P(x) = 3x^4 + 6x^3 - 6x^2 + 24$
1. **Finding composite functions and their domains**  Section 6.1
   
   **Example:** For \( f(x) = \frac{1}{x+3} \) and \( g(x) = \frac{1}{x-2} \), find \( (f \circ g)(x) \) and it’s domain.
   
   **Answer:** \( (f \circ g)(x) = \frac{x-2}{3x-5} \), domain = \( \{x|x \neq \frac{2}{3}, 2\} \)

2. **Finding inverse functions**  Section 6.2
   
   **Example:** For \( f(x) = \frac{1}{3x-2} \), find \( f^{-1}(x) \).
   
   **Answer:** \( f^{-1}(x) = \frac{1+2x}{3x} \)

3. **Graphing exponential functions**  Section 6.3
   
   **Example:** Sketch the graph of \( f(x) = 4 - e^{-x} \).

   **Answer:**

4. **Graphing logarithmic functions**  Section 6.4
   
   **Example:** Sketch the graph of \( f(x) = 3 - \log(x+1) \).

   **Answer:**

5. **Simplifying expressions involving logarithms**  Section 6.5
   
   **Example:** Write as a single logarithm. \( 20\log_2 \sqrt[4]{x} + \log_2 (4x^3) - \log_2 4 \)
   
   **Answer:** \( \log_2 (x^4) \)

6. **Solving logarithmic equations**  Section 6.6
   
   **Example:** Solve. \( \log_{15} x + \log_{15} (x - 2) = 1 \)
   
   **Answer:** \( x = 5 \)

7. **Solving exponential equations**  Section 6.6
   
   **Example:** Solve. \( 2^{x+3} = 5^x \)
Answer: \( x = \frac{-3\ln 2}{\ln 2 - \ln 5} \)

8. **Modeling using exponential functions** Section 6.8

Example: A population of bacteria obeys the law of uninhibited growth. If 600 bacteria are present initially and there are 800 after one hour,

a. Express the population \( P \) as a function of time \( t \).
b. How long will it be until the population doubles? (Write an exact answer.)

Answer: a. \( P(t) = 600e^{\ln(\frac{4}{3})} \)   b. \( \frac{\ln 2}{\ln(\frac{4}{3})} \) hours
1. **Finding the center and radius of a circle**  Section 2.4  
   Example: Find the center and radius of the circle with equation \( x^2 + y^2 - 6x + 10y + 25 = 0 \).
   Answer: center is \((3, -5)\), radius is 3

2. **Graphing parabolas**  Section 7.2  
   Example: For the parabola defined by the equation \( x^2 - 4x = 8y - 28 \), determine the vertex, focus, and directrix and sketch the graph.
   Answer: vertex is \((2, 3)\), focus is \((2, 5)\), directrix is \(y = 1\)

3. **Graphing ellipses**  Section 7.3  
   Example: For the ellipse defined by the equation \( x^2 + 3y^2 - 12y + 9 = 0 \), determine the center, vertices, and foci and sketch the graph.
   Answer: center is \((0, 2)\), vertices are \((3, 2)\) and \((-3, 2)\), foci are \((-\sqrt{2}, 2)\) and \((\sqrt{2}, 2)\)

4. **Graphing hyperbolas**  Section 7.4  
   Example: For the hyperbola defined by the equation \( y^2 - x^2 + 4x - 4y - 1 = 0 \), determine the center, vertices, foci, transverse axis, asymptotes and sketch the graph.
   Answer: center is \((2, 2)\), vertices are \((2, 1)\) and \((2, 3)\), foci are \((2, 2 - \sqrt{2})\) and \((2, 2 + \sqrt{2})\), transverse axis is \(x = 2\), asymptotes are \(y - 2 = x - 2\) and \(y - 2 = -(x - 2)\)
5. **Using the equations of conics to solve applied problems**  Sections 7.2, 7.3, 7.4  

**Example:** A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is to be located. If the dish is 10 feet across at its opening and 6 feet deep at its center, at what position should the receiver be placed?

**Answer:** It should be placed 1 foot and $\frac{1}{2}$ inch from its base along its axis of symmetry.
Essential Skills Chapter 8

1. **Solving systems of linear equations**  Section 8.1

   **Example:** Solve. \[
   \begin{cases}
   .5x + .3y = 2.7 \\
   .7x - .2y = 1.3 
   \end{cases}
   \]

   Answer: \( x = 3, \ y = 4 \)

2. **Solving systems of nonlinear equations**  Section 8.6

   **Example:** Solve. \[
   \begin{cases}
   x^2 + y^2 = 100 \\
   3x - y = 10 
   \end{cases}
   \]

   Answer: \((0, -10)\) and \((6, 8)\)

3. **Using systems of equations to solve applied problems**  Section 8.1

   **Example:** Find real numbers \( a, b, \) and \( c \) so that the function \( y = ax^2 + bx + c \) contains the points \((-1, 6), (2, 8), \) and \((0, 4)\).

   Answer: \( y = \frac{4}{3}x^2 - \frac{2}{3}x + 4 \)