Chapter 9

Key Ideas
Hypothesis Test (Two Populations)

Section 9-1: Overview

In Chapter 8, discussion centered around hypothesis tests for the proportion, mean, and standard deviation/variance of a single population. However, often researchers want to compare two different populations. For example, surgeons may want to try out a new surgical technique for a certain ailment, but they are not sure if it will be better. To test this, they could take two samples of people. In one sample, they could use the existing technique, and in the other, they could use the new technique. By comparing survival proportions of the two groups, they could then determine whether the new sample is any better. In this case, the first population is all people who would receive the existing technique. The second population is all people who would receive the new technique. By using the methods discussed in this chapter, such inference can be done.

Section 9-2: Inferences About Two Proportions

To use the methods described in this section, we first need to rely on a few conditions that must be met for everything to work properly.

Conditions

1. The proportions are taken from two simple random samples which are *independent*. Here, independent means that observations from the first population are not related to, or paired with, observations from the second population.
2. For each of the two samples, there are at least 5 successes and 5 failures.

Notation

p_1 = Population Proportion (Population 1)
n_1 = Sample Size (Population 1)
x_1 = Number of Successes (Population 1)
$\hat{p}_1 = \frac{x_1}{n_1}$ = Sample Proportion (Population 1)
$\hat{q}_1 = 1 - \hat{p}_1$

p_2 = Population Proportion (Population 2)
n_2 = Sample Size (Population 2)
x_2 = Number of Successes (Population 2)
$\hat{p}_2 = \frac{x_2}{n_2}$ = Sample Proportion (Population 2)
$\hat{q}_2 = 1 - \hat{p}_2$

$\overline{p} = \frac{x_1 + x_2}{n_1 + n_2}$ = Pooled Sample Proportion
$\overline{q} = 1 - \overline{p}$

The Test

The goal is to test the hypotheses given by:

H_0: $p_1 = p_2$
H_1: $p_1 \neq p_2$ (or $p_1 > p_2$ or $p_1 < p_2$)

In other words, we test whether the proportions for each population are equal vs. whether they are different in some way.

Test Statistic: $Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\overline{p}\overline{q}}{n_1} + \frac{\overline{p}\overline{q}}{n_2}}}$, or $Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\overline{p}\overline{q}}{n_1} + \frac{\overline{p}\overline{q}}{n_2}}}$ (For the test statistic, we assume H_0 is true, which means $p_1 - p_2 = 0$)

The critical values and P-Values come from the *standard normal distribution*.

Decisions are made in exactly the same way as in Chapter 8.

Example

Close to an election, ballot issues #3 and #4 are very controversial. Researchers want to see whether there is a difference in the proportions of people who support issue #3 and those who support issue #4. They randomly sample 200 total people. They ask 100 of these people whether they support issue #3, to which 56 say "yes". They ask the other 100 people whether they support issue #4, to which 45 say "yes". Is there a difference in the proportions of supporters for each issue? Test this with $\alpha = 0.05$.

Solution

From the information given, we see that:

$n_1 = 100$, $x_1 = 56$, $\hat{p}_1 = 0.56$, $n_2 = 100$, $x_2 = 45$, $\hat{p}_2 = 0.45$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{56+45}{100+100} = \frac{101}{200} = 0.505, \quad \bar{q} = 0.495$$

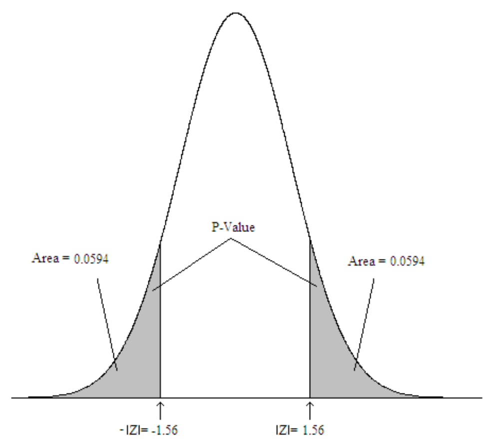
$H_0: p_1 = p_2$

$H_1: p_1 \neq p_2$

Test Statistic: $$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}} = \frac{0.56 - 0.45}{\sqrt{\frac{0.505(0.495)}{100} + \frac{0.505(0.495)}{100}}} = \frac{0.11}{0.0707} = 1.56$$

P-Value Method

- Since H_1 has a "≠" sign, we want to find area above $|Z| = 1.56$ and below $-|Z| = -1.56$ for the standard normal distribution.
- From the Z-Table, this area is 2(0.0594) = 0.1188.
- Now we compare this area to α and see that 0.1188 > 0.05.
- Therefore, we do not reject H_0. In other words, we conclude that there is not enough evidence to claim that there is a difference in proportions of supporters for the two issues.



Example

Gloria, a hairdresser, claims that she is better than a fellow hairdresser named Jules. They decide to run a hypothesis test to see if this is true. Out of 92 of Gloria's customers, 87 are satisfied with their haircut. Out of 67 of Jules' customers, 56 are satisfied. Run the test to see if Gloria's customers have a higher percentage of satisfaction than Jules' customers ($\alpha = 0.05$).

Solution

From the information given, we see that:

$n_1 = 92$, $x_1 = 87$, $\hat{p}_1 = 0.946$, $n_2 = 67$, $x_2 = 56$, $\hat{p}_2 = 0.836$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{87+56}{92+67} = \frac{143}{159} = 0.899, \quad \bar{q} = 0.101$$

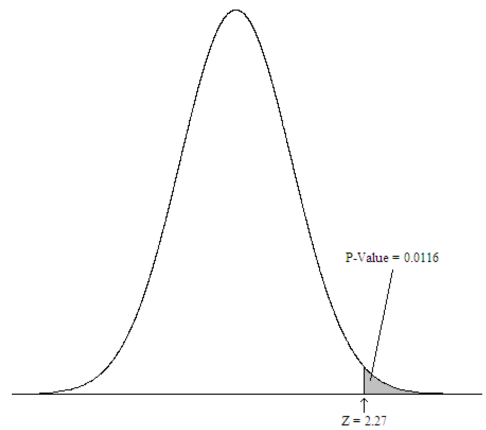
$H_0: p_1 = p_2$

$H_1: p_1 > p_2$

Test Statistic: $$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}} = \frac{0.946 - 0.836}{\sqrt{\frac{0.899(0.101)}{92} + \frac{0.899(0.101)}{67}}} = \frac{0.11}{0.048} = 2.27$$

P-Value Method

- Since H_1 has a ">" sign, we want to find area above Z = 2.27 for the standard normal distribution.
- From the Z-Table, this area is 1 – 0.9884 = 0.0116.
- Now we compare this area to α and see that 0.0116 < 0.05.
- Therefore, we reject H_0 and conclude that Gloria has a higher satisfaction percentage than Jules.



Confidence Interval for $p_1 - p_2$

Sometimes, one would like to estimate the difference between the proportions, rather than just seeing whether the proportions differ significantly. To construct a confidence interval for $p_1 - p_2$, we have the following:

Point Estimate: $\hat{p}_1 - \hat{p}_2$

Critical Value: $Z_{\alpha/2}$

Standard Error: $\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$

This gives the following confidence interval:

$$\hat{p}_1 - \hat{p}_2 \pm Z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

Example

Consider the Gloria/Jules hairdresser example from the previous page.
We had the following quantities:

$n_1 = 92$, $x_1 = 87$, $\hat{p}_1 = 0.946$, $n_2 = 67$, $x_2 = 56$, $\hat{p}_2 = 0.836$

Therefore, a 95% confidence interval would be:

$$\hat{p}_1 - \hat{p}_2 \pm Z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} = 0.946 - 0.836 \pm 1.96\sqrt{\frac{0.946(0.054)}{92} + \frac{0.836(0.164)}{67}}$$

$$= 0.11 \pm 1.96(0.051) = 0.11 \pm 0.09997$$

$$\Rightarrow \boxed{(0.01, 0.21)}$$

Notice that 0 is not included in this confidence interval. Due to this fact, we could say that there is a difference between the two proportions (i.e. we would reject H_0 in favor of H_1: $p_1 \neq p_2$).

Section 9-3: Inferences About Two Means: Independent Samples

In similar fashion to testing for differences in proportions, one may also wish to test for a difference in the means of two populations. In the interests of time, we will consider the most general case, where the population standard deviations are unknown, and no assumptions are made about them. Better hypothesis tests exist when these value are both known, or are unknown but assumed to be equal. Consult the textbook for more information on these tests.

Again, certain requirements must be met for the techniques discussed in this section to be theoretically sound.

Conditions

1. σ_1 and σ_2 are unknown and no assumption is made about the equality of σ_1 and σ_2.
2. The two samples are *independent*.
3. Both samples are simple random samples.
4. Either both populations are normally distributed or n_1 and n_2 are both greater than 30.

Notation

μ_1 = Population Mean (Population 1) μ_2 = Population Mean (Population 2)
n_1 = Sample Size (Population 1) n_2 = Sample Size (Population 2)
$\bar{x}_1$ = Sample Mean (Population 1) $\bar{x}_2$ = Sample Mean (Population 2)
s_1 = Sample Standard Deviation (Population 1) s_2 = Sample Standard Deviation (Population 2)
Degrees of Freedom = df = $\min(n_1, n_2) - 1$
(so df is still $n - 1$, but here n is the smaller of the two sample sizes)

The Test

The goal is to test the hypotheses given by:

H_0: $\mu_1 = \mu_2$
H_1: $\mu_1 \neq \mu_2$ (or $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$)

In other words, we test whether the means for each population are equal vs. whether they are different in some way.

Test Statistic: $$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}, \quad \text{or} \quad t = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

(For the test statistic, we assume H_0 is true, which means $\mu_1 - \mu_2 = 0$)

The critical values and P-Values come from the *Student t distribution* with Degrees of Freedom = $\min(n_1, n_2) - 1$.

Decisions are made in exactly the same way as in Chapter 8.

Example

Researchers want to see if the average money spent per week by tourists in Chicago is less than the average money spent per week by tourists in New York City. They take a sample of 60 tourists from Chicago and 56 tourists from NYC. Of the Chicago tourists, average spending was \$635, with a standard deviation of \$50. Of the NYC tourists, the average spending was \$650, with a standard deviation of \$30. Run a hypothesis test with $\alpha = 0.05$.

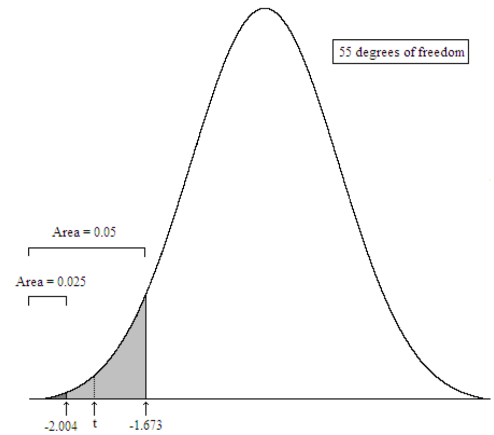
Solution

From the information given, we see that:
$n_1 = 60$, $\bar{x}_1 = 635$, $s_1 = 50$, $n_2 = 56$, $\bar{x}_2 = 650$, $s_2 = 30$, df = $\min(n_1, n_2) - 1 = 56 - 1 = 55$

H_0: $\mu_1 = \mu_2$
H_1: $\mu_1 < \mu_2$

Test Statistic: $$t = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{635 - 650}{\sqrt{\frac{50^2}{60} + \frac{30^2}{56}}} = \frac{-15}{7.599} = -1.974$$

P-Value Method



- Since H_1 has a "<" sign, we want to find area below t = -1.974 for the t distribution.
- Now, notice that the t-Table does not allow one to directly find this area. However, for df = 55 we see that -2.004 has an area below of 0.025 and -1.673 has an area below of 0.05.
- Since $-2.004 < t < -1.673$, the p-value will fall between 0.025 and 0.05. (see picture)
- Now we compare this area to α and see that $0.025 < p < 0.05 = \alpha$.
- Since $p < 0.05$, we reject H_0. The evidence suggests that Chicago tourists pay less per week than those in New York City.

Example

One question on everyone's mind is whether there is a difference in the average number of pets owned by Columbus families and the average number of pets owned by Cleveland families. Researchers set out to answer this important question. They sampled 35 Columbus families and 48 Cleveland families. Of the Columbus families, the average number of pets was 2.4, with a standard deviation of 1.4. Of the Cleveland families, the average number of pets was 1.9, with a standard deviation of 0.9. Run a hypothesis test (α = 0.05) to see if there is a difference in the average number of pets owned by families in the two cities.

Solution

From the information given, we see that:

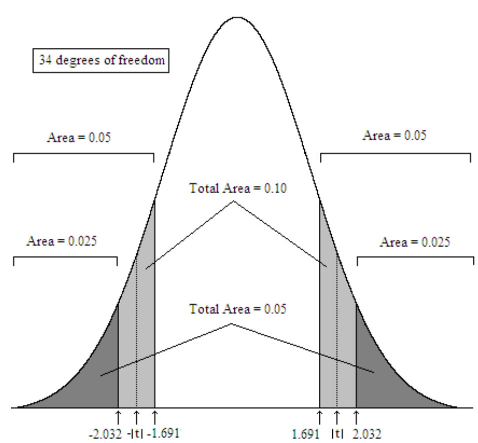
$n_1 = 35$, $\bar{x}_1 = 2.4$, $s_1 = 1.4$, $n_2 = 48$, $\bar{x}_2 = 1.9$, $s_2 = 0.9$, df = min(n_1, n_2) – 1 = 35 – 1 = 34

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

Test Statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{2.4 - 1.9}{\sqrt{\dfrac{1.4^2}{35} + \dfrac{0.9^2}{48}}} = \dfrac{0.5}{0.2700} = 1.852$

P-Value Method



- Since H_1 has a "≠" sign, the p-value will be the area above $|t|$ = 1.852 and below $-|t|$ = -1.852 for the t distribution.
- Again, notice that the t-Table does not allow one to directly find this area. However, for df = 34 we see that 1.691 has a two-tailed area of 0.10 and 2.032 has a two-tailed area of 0.05.
- Since $1.691 < t < 2.032$, the p-value will fall between 0.05 and 0.10. (see picture)
- Now we compare this area to α and see that $p > 0.05 = \alpha$.
- Since $p > 0.05$, we do not reject H_0. There is not sufficient evidence to conclude that the average number of pets is different in the two cities.

Confidence Interval for $p_1 - p_2$

As in the previous section, one might like to estimate the difference between the means, rather than just testing for the difference. To construct a confidence interval for $\mu_1 - \mu_2$, we have the following:

Point Estimate: $\bar{x}_1 - \bar{x}_2$

Critical Value: $t_{\alpha/2}$ (df = min(n_1, n_2) – 1)

Standard Error: $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

This gives the following confidence interval:

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Example

Consider the avg. number of pets example from the previous page.
We had the following quantities:

$n_1 = 35$, $\bar{x}_1 = 2.4$, $s_1 = 1.4$, $n_2 = 48$, $\bar{x}_2 = 1.9$, $s_2 = 0.9$, df = min(n_1, n_2) – 1 = 35 – 1 = 34

Therefore, a 95% confidence interval would be:

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 2.4 - 1.9 \pm 2.032\sqrt{\frac{1.4^2}{35} + \frac{0.9^2}{48}}$$

$$= 0.5 \pm 2.032(0.2700) = 0.5 \pm 0.5485$$

$$\Rightarrow \boxed{(-0.049, 1.049)}$$

Notice that 0 is included in this confidence interval. Due to this fact, we could say that there is not a significant difference between the two means (i.e. we do not reject H_0 in favor of H_1: $\mu_1 \neq \mu_2$, as in the example on the previous page).