

Chapter 4

Key Ideas

Events, Simple Events, Sample Space, Odds, Compound Events, Independence, Conditional Probability, 3 Approaches to Probability Addition Rule, Complement Rule, Multiplication Rule, Law of Total Probability, Tree Diagrams

Section 4-1: Overview

Chapters 1-3 dealt with describing and comparing data sets. Now we move into the realm of inferential statistics. Statistical inference is the science of making educated guesses about population values that cannot be directly measured. In order for the techniques used in inference to make sense, however, we need to study probability, which is the basis for these methods.

Section 4-2: Fundamentals

Definitions:

Event – Any collection of results or outcomes of a procedure (e.g. “I eat a sandwich and drink some water”)

Simple Event – An outcome or event that cannot be broken down into simpler components (e.g. “I eat a sandwich”)

Sample Space – The collection of all simple events that could result from a procedure.

Complement – The complement of an event A consists of all outcomes where A does *not* occur.

Notation:

P – this means “probability”

A, B, C, ... – these stand for events

A^c , \bar{A} – this means the complement of A

$P(A)$ – this means the probability that A occurs

S – this is used to denote the sample space

Example:

Q: Bob rolls a 6-sided die. What is the sample space of this procedure?

A: $S = \{1, 2, 3, 4, 5, 6\}$

Q: Sue measures how many coin flips it takes to get 3 heads. What is the sample space of this procedure?

A: $S = \{3, 4, 5, \dots\} = \{\text{all integers} > 2\}$

Q: Fred sees what proportion of cars on his block are SUVs. What is the sample space of this procedure?

A: $S = \{\text{any real number between 0 and 1}\} = [0,1]$

Since events come in a lot of different ways, there are 3 general approaches to finding the probabilities for events. The method that is most useful depends on the situation.

Approach #1: Relative Frequency Approximation

For procedures that can be repeated over and over again, we can estimate the probability of an event A by using the following:

$$p = \frac{\text{Number of Times A Occurred}}{\text{Total Number of Trials}}$$

From theoretical arguments (see “Law of Large Numbers”), it turns out that this value p will get closer to $P(A)$ as the number of trials gets larger.

Approach #2: Classical Approach

For procedures with equally likely outcomes (e.g. rolling a die, flipping a coin, etc.), we can find $P(A)$ directly, by computing:

$$P(A) = \frac{\text{Number of Ways A can Occur}}{\text{Total Number of Simple Event Outcomes}}$$

Approach #3: Subjective Probability

For procedures that cannot be repeated, and do not have equally likely outcomes, the true probability of an event is usually not able to be determined. In situations like this, we can *estimate* the probability using our knowledge and experience of the subject. For instance, we could ask “What is the probability that the Columbus Blue Jackets will win the Stanley Cup this year?” No one knows the true probability, but people who know a lot about hockey could give a ballpark figure.

Examples:

A situation where Approach #1 is used is in baseball. If we want to know the probability that a player will get a hit when they go up to bat, we cannot use Approach #2 because the outcomes are not equally likely. We could use Approach #3, but that would be subjective. However, by dividing the number of hits by the number of at-bats gives the *batting average*, which is an estimate of the true probability of getting a hit.

A situation where Approach #2 is used is something like rolling a die. Each face is equally likely to turn up, so we can find probabilities using this approach. Let's say A is the event of rolling an even number. What is $P(A)$?

$$P(A) = \frac{\text{Number of Ways A can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{3 \text{ (getting 2, 4, or 6)}}{6 \text{ (6 possible outcomes)}} = \frac{1}{2} = 0.5$$

Another Example

Q: Joe flips one coin 3 times and records the 3 outcomes. What is the sample space?

A: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Q: What is the probability of getting 1 or 2 heads?

A: Since all outcomes are equally likely (we assume the coin is fair), we can use Approach #2.

$$P(A) = \frac{\text{Number of Ways A can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{6 \text{ (HHT, HTH, HTT, THT, TTH, THH)}}{8 \text{ (Total of 8 outcomes)}} = \frac{3}{4} = 0.75$$

Q: What is the probability of the complement of the previous event? (i.e. $P(\text{not 1 or 2 heads})$)

A: There are 2 ways this can happen (HHH or TTT), so it will be $\frac{2}{8} = \frac{1}{4} = 0.25$

Section 4-3: Addition Rule

Compound Event – An event that is comprised of two or more *simple events*.

Generally, compound events are written in terms of their simple events.

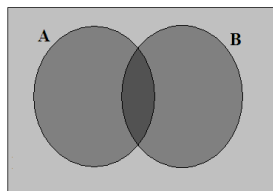
For example, the event “It will rain or snow today” could be written as “**A or B**”, where A is the event that it rains today and B is the event that it snows today. So this event would happen if it rained today, snowed today, or both.

Another type of event is of the form “**A and B**”, in which the event only occurs if *both* A and B occur.

For example, the event “It is at least 70 degrees and sunny outside” could be written as “A and B”, where A is the event that it is at least 70 degrees outside, and B is the event that it is sunny.

The Formal Addition Rule

To see how this rule is derived, let's examine a *Venn Diagram*. The area within each circle corresponds to the probability of that event occurring. Where the two circles overlap (dark grey), both A and B occur. However the area, say, in circle A that does not overlap B (grey) would be when A occurs but B does not. The area outside of both circles (light grey) corresponds to neither A nor B occurring.



How would we find $P(A \text{ or } B)$ then? We want the area within the two circles (the grey and dark grey areas) because that's where A happens, B happens, or they both happen. What we could do is add together the area in circle A, and the area of circle B. The problem is that we could the overlapping area (dark grey) twice. That means we need to subtract it. Using the fact that the area in circle A is $P(A)$, the area in circle B is $P(B)$, and the overlap is $P(A \text{ and } B)$, we get the formal addition rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This rule works for any events A and B. Anytime you know three of the four quantities in the equation, you can solve for the fourth.

Disjoint Events

Disjoint Events are events that cannot both happen at the same time. For example, let A be the event that a traffic light is green, and B be the event that the traffic light is red. The event “A and B” cannot happen, because traffic lights are never green and red at the same time. If two events are disjoint, then $P(A \text{ and } B) = 0$. Disjoint events are also often called *mutually exclusive events*.

Addition Rule for Disjoint Events

Using the fact that $P(A \text{ and } B) = 0$ for disjoint events, we can rewrite the formal addition rule as:

$$P(A \text{ or } B) = P(A) + P(B)$$

Complementary Events

Recall from the previous section that for an event A , its complement A^c is the event that A does *not* occur. Since A^c only happens when A does not, and vice versa, $P(A \text{ and } A^c) = 0$. In other words, A and A^c are disjoint. Therefore, $P(A \text{ or } A^c) = P(A) + P(A^c)$, by the addition rule for disjoint events. But what is $P(A \text{ or } A^c)$? This means the probability that either A happens, or A does not happen. This probability is 1, since *something* has to happen, whether it is A or not. Therefore, we have a trio of equivalent formulas:

$$P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

Examples: For the following questions, imagine we are drawing one card from a deck of 52 cards.

Q: What is the probability of drawing a queen?

A: Using Approach #2 from the previous section, and letting A be the event in question,

$$P(A) = \frac{\text{Number of Ways } A \text{ can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{4}{52} = \frac{1}{13}$$

Q: What is the probability of drawing a diamond?

A: Using Approach #2 from the previous section, and letting B be the event in question,

$$P(B) = \frac{\text{Number of Ways } B \text{ can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{13}{52} = \frac{1}{4}$$

Q: What is the probability of drawing a queen of diamonds?

A: This is the event " A and B ", and we can use Approach #2 again:

$$P(A \text{ and } B) = \frac{\text{Number of Ways } A \text{ and } B \text{ can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{1}{52}$$

Q: What is the probability of drawing a queen or a diamond?

A: We could up the total number of cards that fit this bill (13 diamonds – one of which is a queen – and the other 3 queens = 16 possible cards out of 52), or we can just use the formal addition rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Q: What is the probability of drawing a number card? (Aces included)

A: Let's call this event C . There are 4 of each number 1 to 10, for a total of 40 out of 52 cards.

$$P(C) = \frac{\text{Number of Ways } C \text{ can Occur}}{\text{Total Number of Simple Event Outcomes}} = \frac{40}{52} = \frac{10}{13}$$

Q: What is the probability of drawing a number card or a queen?

A: Now we can use the addition rule for disjoint events, since A and C can't happen at the same time.

$$P(A \text{ or } C) = P(A) + P(C) = \frac{4}{52} + \frac{40}{52} = \frac{44}{52} = \frac{11}{13}$$

Section 4-4, 4-5: Multiplication Rule: The Basics

Now that we can find " A or B " probabilities, we focus on how to find " A and B " probabilities. Intuitively, for A and B to happen, we need two things to take place:

- i. A needs to happen
- ii. Given that A happened, B needs to happen

This leads us to...

The Formal Multiplication Rule

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Here, $P(B|A)$ is what is called a *conditional probability*. It stands for the probability that B happens *given* A already happened. (The vertical bar means “given”) Since “A and B” is the same as “B and A”, we can also write the formula as:

$$P(A \text{ and } B) = P(B) \cdot P(A|B)$$

Which form you use depends on what information is available.

Example: Students can take a standardized test at three test centers A, B, and C. Suppose that after the most recent test, 500 students went to A, 200 went to B, and 300 went to C. Furthermore, the proportion of students who passed the exam were 50%, 80%, and 75%, respectively.

Q: What is the probability that a randomly selected student took the test at center B?

A: There are a total of 1000 students, and 200 went to B. Thus $P(B) = 200/1000 = 0.20$.

Q: What is the probability that a student who took the test at B passed the exam?

A: Now, we want to find the probability of passing *given* that the student took the test at B. We are told in the problem that 80% of the students at center B passed. Thus we have: $P(\text{Pass} | B) = 0.80$.

Q: What is the probability that a student both took the test at B and passed?

A: Using the Multiplication Rule, $P(\text{Pass and } B) = P(B)P(\text{Pass} | B) = (0.20)(0.80) = 0.16$.

Independent Events

Two events are called **independent** if the occurrence of one does not affect the chances of the other one occurring. Statistically, what this means is that A and B independent $\rightarrow P(A | B) = P(A)$. In other words, the probability of A happening *given* that B happened is just the same as if we didn't know whether B happened (because the occurrence of B has no effect on the occurrence of A).

Note: if A and B are disjoint, then we know that only one can occur. Thus, knowing that B happened tells you that A definitely did not happen, and we have $P(A | B) \neq P(A)$. Thus disjoint events are *never* independent events.

Multiplication Rule for Independent Events

Using the fact that $P(A | B) = P(A)$ for independent events, we see that the formal multiplication rule turns into:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Section 4-7: Counting

Fundamental Counting Rule: For a sequence of events in which the first event can occur m ways, the second event can occur n ways, the third event can occur p ways, etc, the events together can occur a total of $m \cdot n \cdot p \dots$

Example:

A deli has a lunch special which consists of a sandwich, soup, dessert and drink for \$4.99. They offer the following choices:

Sandwich: chicken salad, ham, and tuna, and roast beef

Soup: tomato, chicken noodle, vegetable

Dessert: cookie and pie

Drink: tea, coffee, coke, diet coke and sprite

How many lunch specials are there?

Let's use the basic counting principle:

There are 4 stages or events: choosing a sandwich, choosing a soup, choosing a dessert and choosing a drink.

There are 4 choices for the sandwich, 3 choices for the soup, 2 choices for the dessert and 5 choices for the drink.

Putting that all together we get: $4 \cdot 3 \cdot 2 \cdot 5 = 120$

So there are 120 different lunch special combinations available.

Notation: The factorial symbol: ! It denotes the product of decreasing positive whole numbers. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. By definition, $0! = 1$

Factorial Rule: n different items can be arranged $n!$ ways.

Example: Six horses are entered in a race. In how many different orders can the horses complete the race?

There are six steps in ordering the horses. First we choose a first place winner, then a second place finisher, then third, fourth, fifth, and sixth. There are 6 choices for first, then 5 choices for second, 4 for third, etc.; therefore, the number of

possible permutations is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 .$$

Permutation: ${}_nP_r$

When n different items are available, and you want to select r of the items (without replacement), and order DOES matter (that is ABC is not the same as BCA). Then the number of permutations is ${}_nP_r = \frac{n!}{(n-r)!}$

Example: A football team has 5 players trying out for the quarterback position. How many ways can the coach select a starting quarterback and a backup quarterback?

Here we are choosing 2 ($r=2$) objects from 5 ($n=5$) objects, and then ordering the two objects - one as a starter and the second as a backup.

$${}_nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20$$

There are 20 ways the coach can select a starting quarterback and a backup quarterback.

Permutation (when some items are identical):

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

where n is the number of items, and there are n_1 alike, n_2 alike,

Example: The word *bookkeeper* is unusual in that it has three consecutive double letters. How many permutations are there of the letters in *bookkeeper*?

If each letter were different, there would be $10!$ permutations (since there are 10 letters in the word), but there are two *os*, two *ks*, and three *es*. You must divide by $2!$ twice to allow for the duplication of the *os* and *ks* and then divide by $3!$ to allow for the three *es*:

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!} = \frac{10!}{2!2!3!} = 151200$$

There are 151 200 permutations of the letters in bookkeeper.

Combination: ${}_nC_r$

When n different items are available, and you want to select r of the items (without replacement), but order does NOT matter (that is ABC is the same as BCA or CBA...). Then the number of combinations is ${}_nC_r = \frac{n!}{(n-r)!r!}$

Example: Six ($n=6$) young women have applied for a part-time job at Miguel's Restaurant. How many ways can Miguel choose three ($r=3$) waitresses?

$${}_nC_r = \frac{n!}{(n-r)!r!} = \frac{6!}{(6-3)!3!} = 20$$

He can choose 20 different ways.