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Radiation-Controlling Boundary Conditions for a Problem of Constrained Evolution

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Formulations of Einstein's equations used in numerical relativity comprise two kinds of equations: the part that contains both temporal and spatial derivatives (called the evolution part) and the part that has only spatial derivatives (called the constraint part). In the absence of boundaries, the evolution equations of general relativity preserve the constraint equations. Namely, if a solution is constructed to the evolution equations in the entire space that satisfies the constraint equations initially, it will automatically satisfy the constraint equations for all times. In practical computations, however, one needs to limit the size of the computational domain, cut out singularities, or to achieve parallelization. This requires introducing artificial boundaries, and as the result, the boundary conditions.

Boundary conditions have a dramatic impact on the properties of the system and must be carefully chosen to avoid solving an ill-posed problem. It turns out that in numerical relativity the intuition that can be acquired by examining the evolution part along is misleading: the most standard choices of maximally dissipative boundary conditions (which are well-posed for the evolution equations) result in constraint violations.

A hint on what is happening there can be obtained by examining the evolution of constraint quantities. A subsidiary system of equations can be derived from the evolution part of the system, e.g., in the linearized regime, that describe propagation of the constraint quantities. Properties of the subsidiary system can be studied to find classes of boundary conditions for the main system that are compatible with the constraint preservation. Methods were proposed to construct inhomogeneous mixed Dirichlet - Neumann boundary conditions that imply constraint preservation, see e.g., [3,4,5,7]. The Dirichlet and Neumann data, however, is known to produce large spurious reflections at the boundary and therefore, methods for constructing boundary conditions that control radiation through the boundary and the transparent boundary conditions are in interest. Examples of the boundary conditions that controll radiation were obtained in e.g., [1,6,8,11,12,13,14]. The problem here is that the boundary conditions that one can conceive from the examining the evolution system for constraint propagation, are not in any standard form to which standard well-posedness result apply. The proposed conditions include derivatives tangential to the boundary and higher derivatives, therefore the energy estimates on the boundary data are hard to obtain. The most powerful technique to deal with this kind of boundary conditions involves pseudo-differential reduction using Laplace-Fourier transform, see e.g., [9,11,13,14]. In some cases, the energy methods can be applied in a sophisticated way to prove problem's well-posedness [10].

In this work a new technique [1] (a closely related treatment is used in [6]) is employed for constructing boundary conditions for the model problem of vector wave equation subject to the divergence free constraint. We use static constrain evolution reduction to derive a new set of constraint-compatible boundary condition and prove well-posedness of the resulting problem.

1. Vector wave equation

Consider a vector wave equation in a polyhedral domain Ω,

$$\partial_t^2 u_i = \partial^j \partial_j u_i, \tag{1}$$

(indices are raised and lowered using the flat metric) subject to the divergence-free constraint

$$C := \partial^i u_i = 0. \tag{2}$$

It can be easily verified that the subsidiary evolution of the constraint quantity C is given by the wave equation

$$\partial_t^2 C = \partial^i \partial_i C. \tag{3}$$

Thus, if $C \neq 0$ at some point of the boundary, in view of (3), the constraint perturbation will flow inside the domain. In general, to ensure $C \equiv 0$, the boundary data on u_i must be given consistently to imply the homogeneous data for the constraint quantity. The data that guarantees that C remains zero for all times is called *constraint-preserving*.

Several approaches have been introduced to enforce compatibility of the boundary data based on the fact that if the boundary data is enforcing either of the homogeneous conditions

$$C|_{\partial\Omega} = 0, \quad \partial_n C|_{\partial\Omega} = 0, \quad \text{or}$$

$$(\partial_t C + \partial_n C)|_{\partial\Omega} = 0, \tag{4}$$

then the constraint quantity is guaranteed to remain zero during the evolution, see e.g., [3,4,5,7,13,14,11,1]. The difficulty in these approaches is that after C is replaced in (4) with its definition, one obtains a rather sophisticated expression involving tangential, temporal and, sometimes, higher order derivaitves. In some cases, to ensure compatibility of all terms, an elaborate integration along the boundary is performed to produce compatible inhomogeneous data. Then a set of constraint-preserving boundary conditions can be given in one the standard forms, usually a mixture of inhomogeneous Neumann and Dirichlet conditions [3,4,5,7]. In other cases, the boundary conditions on certain incoming modes are replaced with the analogs of (4) above, and Laplace-Fourier techniques used to prove well-posedness of the resulting problem [7,8,12,13]. It is, however, not always clear how to justify the selection of the modes that will be replaced.

Instead of studying directly (1), (2), we propose to introduce a term $-\partial_i C$ in the right side of (1) and consider the auxiliary equation

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j. \tag{5}$$

It is straightforward to verify that (5) implies the equation

$$\partial_t^2 C = 0. \tag{6}$$

This means that equation (5) still preserves the constraint (2). However, in contrast to (1), in the new system, constraint preservation property in independent of the boundary conditions.

2. The constraint-preserving boundary conditions

Because preservation on the constraint in system (5), (2) depends exclusively on the initial data and is independent of the boundary data, the following result holds.

Theorem 1. Let the 2×2 matrix α^A_B be such that $\|\alpha\| \leq 1$, the field g_A allows an H^1 extension inside Ω (and satisfies some additional compatibility properties described in Theorem 4.8 in [1]). Then the constrained evolution problem (5), (2) has a unique solution satisfying boundary conditions (here u_n and u_A stand for the components of u_i that are normal and tangential to the boundary, respectively)

$$\begin{aligned}(\partial_t u_A + \partial_n u_A - \partial_A u_n) \\ = \quad \alpha_A^B(\partial_t u_B - \partial_n u_B + \partial_B u_n) + g_A, \quad \|\alpha\| \leq 1\end{aligned} \tag{7}$$

provided that the initial data satisfies the constraint (2). In addition, the solution satisfies the energy estimate (here $\partial_{[j}u_{i]} = (\partial_j u_i - \partial_i u_j)/2$ denotes the anti-symmetric part of the gradient)

$$\begin{aligned}\sup_{0\leq t\leq T} [\|\partial_t u_i\|_{L^2(\Omega)} + 2\|\partial_{[j}u_{i]}\|_{L^2(\Omega)}] \\ \leq c\int_0^T \|g\|_{L^2(\partial\Omega)}dt + [\|\partial_t u_i(0)\|_{L^2(\Omega)} + 2\|\partial_{[j}u_{i]}(0)\|_{L^2(\Omega)}].\end{aligned}$$

It is often advantageous from the numerical point of view to solve the free evolution problem versus the fully constrained problem. We recall that only the evolution equation (1) is solved in the free evolution problem, while constraint (2) is monitored but not actively enforced on the solution. The following theorem gives an example of radiation-controlling boundary data that guarantee that the constraint (2) is preserved during the free evolution.

Theorem 2. Let $\alpha \in R$, $|\alpha| \leq 1$, the fields g and g_A are defined on the boundary, are compatible at corners, g_a satisfies the assumptions of the Theorem 1 and both g and g_A satisfy the compatibility conditions

$$\partial_t g = -\partial^A g_A, \quad g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0) \quad \text{on} \quad \partial\Omega. \tag{8}$$

If, in addition, the initial data satisfies the constraint (2), then, the free evolution problem (1) has a unique solution satisfying boundary conditions

$$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A, \tag{9}$$

$$(\partial_t u_n + \partial_n u_n) = -\alpha(\partial_t u_n - \partial_n u_n) + g. \tag{10}$$

Moreover, the energy estimate of Theorem 1 holds.

It is not practical, however, to assume that either (2) is satisfied exactly by the discrete initial data, or that the compatibility conditions (8) hold for the discretized solution. The following theorem establishes the well-posedness of the free evolution problem (1), (9), (10)

Theorem 3. Let $\alpha \in R$, $|\alpha| \le 1$, the fields g and g_A are defined on the boundary $\partial\Omega$ and are compatible at corners. Let the field g_A meets the assumptions of Theorem 1. Let, in addition, the initial and boundary data are compatible:

$$g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0) \quad \text{on} \quad \partial\Omega.$$

Then a unique solution $u_i \in L_2(\Omega)$, $\partial_{[i} u_{j]} \in L_2(\Omega)$, $\partial^i u_i \in L_2(\Omega)$ exists to (1) satisfying boundary conditions (9), (10). Moreover, the solution satisfies the estimate

$$\begin{aligned}
&\sup_{0\le t\le T} [2\|\partial_t u_i\|_{L^2(\Omega)} + \|\partial_{[j} u_{i]}\|_{L^2(\Omega)} + \|\partial^i u_i\|_{L_2(\Omega)}] \\
&\le \quad c\int_0^T \|g_A\|_{H^{1/2}(\partial\Omega)} dt + c_2 \mathrm{e}^T \Big(c_1 \int_0^T \|\partial_t g + \partial^A g_A\|_{H^{1/2}(\partial\Omega)} dt \\
&\qquad + [\|\partial^l \partial_t u_l(0)\|_{L_2(\Omega)} + \|\partial^l u_l(0)\|_{L_2(\Omega)}] \Big) \\
&\qquad\qquad + [2\|\partial_t u_i(0)\|_{L_2(\Omega)} + \|\partial_{[j} u_{i]}(0)\|_{L_2(\Omega)}].
\end{aligned} \tag{11}$$

3. The constraint-preserving boundary conditions

In should be noted that conditions (9), (10) in no sense are in the maximal dissipative form for (1) (they, however, are for (5)). As the result, in spite of the estimate of Theorem 3, there is still a chance that a conventional discritization of (1) will result in an ill-posed problem. In [1] Laplace-Fourier analysis is used to prove that the (1), (9), (10) is well-posed in the generalized sense (see, e.g., [9,11,13,14]), which suggests that the problem can be still solved using standard discretizations.

Following the general procedure, (1) is reduced to first order in the way compatible with the most standard discretization of the wave operator:

$$\begin{aligned} \partial_t u_i &= \pi_i, \qquad \varphi_{ki} = \partial_k u_i \\ \partial_t \pi_i &= \partial^j \varphi_{ji} \\ \partial_t \varphi_{ji} &= \partial_j \pi_i \end{aligned}$$

We seek a solution as a superposition of modes

$$\begin{pmatrix} \pi_i \\ \varphi_{ji} \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_i(x_1) \\ \tilde{\varphi}_{ji}(x_1) \end{pmatrix} \exp(st + \imath\omega_A x^A),$$

where $s \in \mathbb{C}$, $\Re(s) > 0$, ω_A is a real two-vector.

The Kreiss condition is satisfied if fields $\tilde{\pi}_i(0)$, $\tilde{\varphi}_{xi}(0)$ can be bounded by the data. It is shown in [1] that the Kreiss condition is satisfied for (9), (10), namely

$$\|(\tilde{\pi}_i(0), \tilde{\varphi}_{xi}(0))\| \leq K\|g, g_A\|$$

It is shown in [1] that the Kreiss condition is satisfied for (9), (10).

Note that only the most directly related papers are cited below. The full list of citations is available from the author.

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