1. Suppose an approximation \( \hat{y} \) to \( y = f(x) \) needs to be computed, where \( f(x) \) is a scalar function of a scalar variable.

   a) (7pt) Give the definition of the backward error for \( y = f(x) \).

   b) (8pt) If \( f(x) = \frac{1}{x} \), and our calculations for \( y = f(3) \) give \( \hat{y} = 0.3 \). What is the backward error involved in this calculation?
2. Using the backward error result for the vector inner product from Section 3.1 (formula (3.4)): for \( x, y \in \mathbb{R}^n \),

\[
fl(x^T y) = (x + \Delta x)^T y = x^T (y + \Delta y), \quad |\Delta x| \leq \gamma_n |x|, \quad |\Delta y| \leq \gamma_n |y|,
\]
where \( |x| \) denotes the vector with elements \(|x_i|\) and inequalities between vectors hold componentwise;

a) (10pts.) prove the forward error bound (3.5):

\[
|x^T y - fl(x^T y)| \leq \gamma_n |x|^T |y|.
\]

b) (10pts.) prove the following backward error result for matrix-vector product. Let \( A \in \mathbb{R}^{m \times n} \), \( x \in \mathbb{R}^n \), \( y = Ax \), then (see formula (3.11))

\[
\hat{y} = (A + \Delta A)x, \quad |\Delta A| \leq \gamma_0 |A|
\]
3. (15pts.) Using the definition of $\gamma_n$ from Section 3.1:

$$\gamma_n := \frac{nu}{1 - nu},$$

where $u$ is the unit roundoff, $nu \ll 1$. Prove that

$$u \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n.$$
4. (20pts.) Recall that
\[ \|A\|_{\alpha,\beta} = \max_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}. \]

Prove that for any subordinate norm \( \| \cdot \|_\gamma \), the following inequality holds (see formula (6.7))
\[ \|AB\|_{\alpha,\beta} \leq \|A\|_{\gamma,\beta} \|B\|_{\alpha,\gamma}. \]

Hint. One way to prove this is to use the trivial identity
\[ \frac{\|ABx\|_\beta}{\|x\|_\alpha} = \frac{\|ABx\|_\beta}{\|Bx\|_\gamma} \frac{\|Bx\|_\gamma}{\|x\|_\alpha}. \]
5. (15pts.)
The quantity $y = n!$ is computed using the following algorithm:

$s = 1$
for $i = 2 : n$
  $s = s \times i$
end

Write this algorithm using the general framework of Section 3.8, for $n = 3$. Make sure to specify what is function $g_k(x_k)$, what are the components $x_k$, and what form does the matrix $\bar{I}$ takes in this particular example.
6. (15pts.) Let a real vector \( x = (1/2, \sqrt{3}/2) \) be given. Find a real vector \( y \) dual to \( x \) in the \( \| \cdot \|_2 \)-norm.

Recall that \( y \) is the dual vector to \( x \) in the 2-norm, if

\[
y^T x = \| y \|_2^D \| x \|_2 = 1.
\]

and the dual norm is defined by

\[
\| y \|_2^D = \max_{x \neq 0} \frac{\| y^T x \|_2}{\| x \|_2} = \max_{\| x \|_2 = 1} \| y^T x \|_2
\]