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Math 481a Final. December 12, 2005

Attention! Please, note that this is the closed book test. You are NOT allowed to use a graphing calculator. In multiple choice, guessing is not penalized. In other problems, please, show all important steps in you solution.

1a. (5pt) In Horner's method, which of the following is the correct formula for calculating the value of derivative for polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ at a point x_0?

Horner's method: *Let* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$. *If* $b_n = a_n$ *and*

$$b_k = a_k + b_{k+1}x_0, \qquad \text{for } k = n-1, n-2, \ldots, 1, 0.$$

then $b_0 = P(x_0)$. *Moreover, if* $Q(x) = b_n x^{n-1} + b_{n-1}x^{n-1} + \ldots + b_1$, *then* $P(x) = (x - x_0)Q(x) + b_0$.

- ☐ $P'(x_0) = (b_0)'$;
- ☐ $P'(x_0) = (x - x_0)$;
- ☐ $P'(x_0) = Q'(x_0)$;
- ☐ $P'(x_0) = P(x - x_0) - P(x_0)$;
- ☐ $P'(x_0) = Q(x_0)$.

1b. (5pt) Which from the following conditions is true for the Clamped Cubic Spline $S(x)$ approximating function $f(x)$ on interval $[a, b]$.

- ☐ $S'(a) = S'(b) = 0$;
- ☐ $S'(a) = f'(a)$ and $S''(a) = f''(a)$;
- ☐ $S''(a) = S''(b) = 0$;
- ☐ $S'(a) = f'(a)$ and $S'(b) = f'(b)$;
- ☐ none of the above.

1c. (5pt) Hermite polynomial $P(x)$ approximating function $f(x) \in C^2[a,b]$ on the nodes $x_0 = a, x_1, x_2, \ldots, x_n = b$ is the polynomial of least possible degree with properties:

☐ $\dfrac{dP(x_i)}{dx} = \dfrac{df(x_i)}{dx}$, for each $i = 0, 1, 2, \ldots, n$;

☐ $P(x_i) = f(x_i)$ and $\dfrac{dP(x_i)}{dx} = \dfrac{df(x_i)}{dx}$, for each $i = 0, 1, 2, \ldots, n$;

☐ $P(x_i) = f(x_i)$, for each $i = 0, 1, 2, \ldots, n$;

☐ $P(x_i) = f(x_i)$, $\dfrac{dP(x_i)}{dx} = \dfrac{df(x_i)}{dx}$, and $\dfrac{d^2P(x_i)}{dx^2} = \dfrac{d^2f(x_i)}{dx^2}$ for each $i = 0, 1, 2, \ldots, n$;

☐ $\dfrac{dP(x_i)}{dx} = \dfrac{df(x_i)}{dx}$ and $\dfrac{d^2P(x_i)}{dx^2} = \dfrac{d^2f(x_i)}{dx^2}$ for each $i = 0, 1, 2, \ldots, n$.

1d. (5pt) Which from the following correctly completes the definition of the Legendre polynomials?

Legendre Polynomials: *A collection of polynomials* $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots\}$ *with properties: for each* n, $P_n(x)$ *is a monic polynomial of degree* n, *and*

☐ $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n;

☐ $\int_{-1}^{1} P(x)P_n(x)dx \neq 0$ whenever $P(x)$ is a polynomial of degree less than n;

☐ $\int_{-1}^{1} P_n(x)P_n(x)dx = 0$;

☐ $P_n(x)$ has n distinct roots for each value of n;

☐ $P_n(x)$ has n distinct roots on $[-1, 1]$ for each value of n;

2. (10pt) Prove that the sequence

$$\alpha_n = \left(\sin\frac{1}{n}\right)^5$$

converges to zero with a rate similar to the sequence $\{\frac{1}{n^5}\}$

(10pt) On **what interval** the bisection method has to be applied to the equation

$$x = \frac{\pi}{2}\sin x$$

to approximate the root $x = \frac{\pi}{2}$.

3. (7pt) The equation

$$x = \frac{1}{e^x}$$

is solved on the interval $[0, 1]$ by the **fixed-point** method. In the fixed point method, $p_{n+1} = g(p_n)$, decide which one of the two functions

$$g(x) = \frac{1}{e^x} \quad \text{or} \quad g(x) = -\ln x.$$

must be used. Justify your answer.

(8pt) Use the correct function and Corollary 2.4 to estimate the number of iterations required to compute the fixed point with 10^{-5} accuracy from the initial approximation $p_0 = 0.5$.

4. (5pt) State the definition of the zeroth, first, and second divided differences of function $f(x)$ relative to nodes x_1, x_2, x_3.

(10pt) Let the function be interpolated on the nodes $x_0 = 0$, $x_1 = b$, $x_2 = 1$. Find value of b, if it is known that the second divided difference of the polynomial $P(x) = x^2(x-1)$ on the nodes x_0, x_1, x_2 is equal to $1/3$.

5. (5pt) Write the definition of the composite Simpson's rule for approximating the integral

$$\int_a^b f(x)dx.$$

(10pt) Use Theorem 4.4 to estimate the number of nodes necessary to approximate the integral

$$\int_0^3 xe^x dx$$

by the composite Simpson's Rule with 10^{-5} accuracy:

Theorem 4.4. *Let $f(x) \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule S_n for n subintervals can be written with its error term as*

$$\int_a^b f(x)dx = S_n - \frac{b-a}{180} h^4 f^{(4)}(\mu).$$

6. Consider the initial value problem problem

$$y'(t) = \mathrm{e}^t y, \qquad 0 \le t \le 1, \qquad y(0) = 1.$$

(8pt) Does the function $f(t, y) = \mathrm{e}^t y$ satisfy the Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$? If yes, what is the corresponding Lipschitz constant?

(7pt) Formulate the Euler method to approximate solution to the initial value problem

$$y'(t) = \mathrm{e}^t y, \qquad 0 \le t \le 1, \qquad y(0) = 1.$$