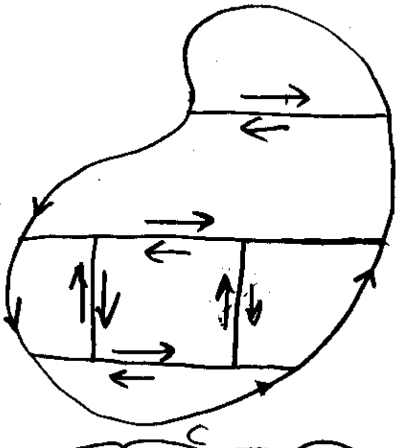
# Green's Theorem
## In the plane



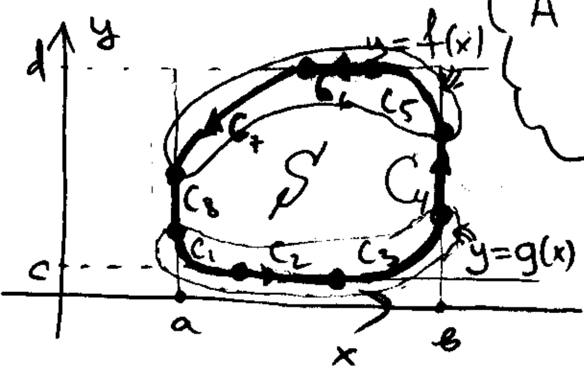

**Thm A** Let $C$ be a piecewise smooth, simple closed curve that forms the boundary of a region $S$ in XY-plane. If $M(x,y)$ and $N(x,y)$ are continuous and have continuous partial derivatives on $S$ and its boundary $C$, then

$$\int_a^b F'(y)\,dy = F(b) - F(a)$$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy$$

**Proof** First prove the statement for a domain of the type

A generic domain CAN BE splitted into simple ones like on the example above




Consider the integral over the closed curve $C$:

$$\oint_C M\,dx = \int_{C_1\cup C_2\cup C_3} M\,dx + \cancelto{0}{\int_{C_4} M\,dx} + \int_{C_5\cup C_6\cup C_7} M\,dx + \cancelto{0}{\int_{C_8} M\,dx}$$

$$= \int_a^b M(x, g(x))\,dx + \int_b^a M(x, f(x))\,dx = -\int_a^b \left[M(x, f(x)) - M(x, g(x))\right] dx$$

$$= -\int_a^b \left[\int_{g(x)}^{f(x)} \frac{\partial M}{\partial y}(x,y)\,dy\right] dx = -\iint_S \frac{\partial M}{\partial y}(x,y)\,dA$$

Similarly,

$$\oint_C N\,dy = \iint_S \frac{\partial N}{\partial x}\,dA$$