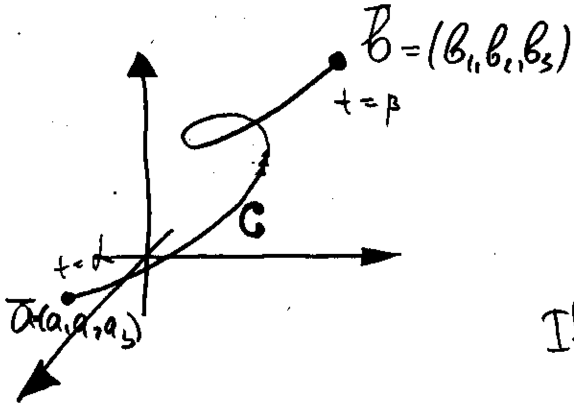
Indepence of Path



Let C be a piecewise smooth curve given parametrically by

$$\bar r=\bar r(t)=(x(t),y(t),z(t)),\quad \alpha\le t\le\beta$$

$$\bar a=\bar r(\alpha),\quad \bar b=\bar r(\beta)$$

If f is continuously differentiable on an open set containing C, then

$$\int_C \nabla f(r)\cdot dr = f(b)-f(a) \quad (*)$$

Notice that the right side of (*) depends only on the end points.
Does this theorem imply that the result will be the same for any path?

Independence of Path Thm. Let F(r) be continuous on an open connected set D

connected means any two points can be connected by a curve laying interely in D:

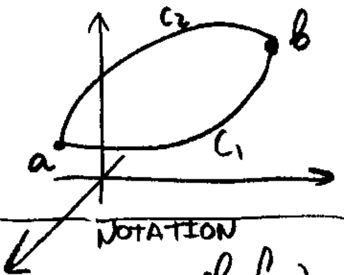
Then the line integral

$$\int_C \bar F(r)\cdot d\bar r \text{ is independent of Path}$$

if and only if

$$\bar F(r)=\bar\nabla f \text{ for some scalar function } f$$

(such $\bar F$ is called a conservative vector field)



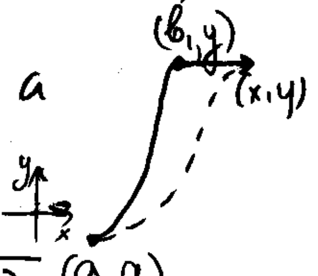
NOTATION

$$\int_C F(\bar r)\,d\bar r=\int_{(a_1,a_2)}^{(b_1,b_2)} \bar F(r)\cdot dr$$

Proof ⇐ follows from the above statement
⇒ let us construct a function satisfying $\bar\nabla f=\bar F$ (Consider 2-D case)

Let $f(x,y)$ = Integral of $\int_C \bar F\cdot d\bar r$ between $a=(a_1,a_2)$ and (x,y); Choose a special path

$$f(x,y)=\int_C \bar F\cdot d\bar r = \text{independent of path} = \int_{(a_1,a_2)}^{(b_1,y)} \bar F\cdot dr + \int_{(b_1,y)}^{(x,y)} \bar F\, dr$$



$d\bar r=(dx,0)$

$$\bar F\cdot d\bar r = F_x dx + \underbrace{F_y\cdot 0}_{\to 0}$$

$$\int_{(b_1,y)}^{(x,y)} \bar F\cdot d\bar r = \int_b^x F_x(t,y)\,dt$$

Check

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left[\cancel{\int_{(a_1,a_2)}^{(b_1,y)} F\cdot dr}\right]_{\to 0}+\frac{\partial}{\partial x}\left[\int_b^x F_x(t,y)\,dy\right]=F_x(x,y)$$

Similarly, $\frac{\partial f}{\partial y}=F_y(x,y)$