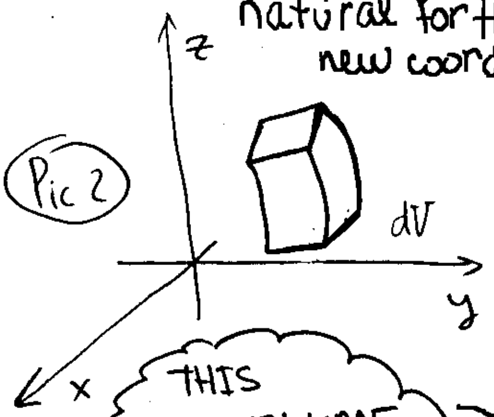
CHANGE OF VARIABLES in triple integrals.

the partitioning natural for the new coordinates

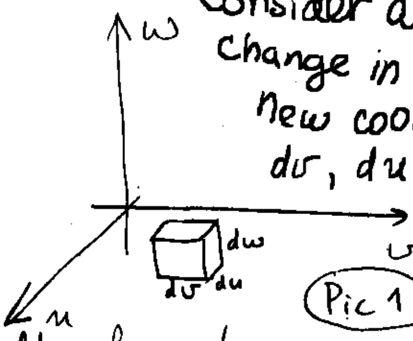


Pic 2

$(x,y,z) \longrightarrow (u,v,w)$

$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$

Consider a small change in the new coordinates: dv, du, dw



Pic 1

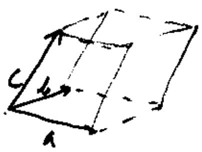
THIS VOLUME ELEMENT corresponds to change $dv\,du\,dw$ in new coordinates

To write change of variables formula we need expression for the volume dV in terms of new variables

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_{S'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot ???$$

Here is a good formula for calculating volumes:

Volume of parallelepiped built on vectors



$a = (a_1, a_2, a_3)$
$b = (b_1, b_2, b_3)$
$C = (c_1, c_2, c_3)$

$$V = |a\cdot(b\times c)| = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|$$

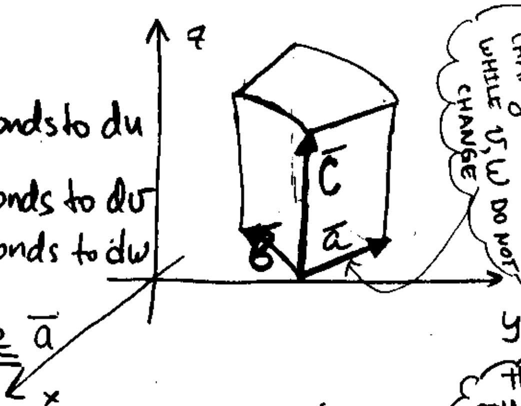
absolute value of the determinant of the matrix

Let's use it in our stuff

Let us calculate the expression for dV:

We start from finding vectors that make up the sides of dV

Assume:
$\overline{a}$ corresponds to du
$\overline{b}$ corresponds to dv
$\overline{c}$ corresponds to dw



corresponds to change in u, while v, w do not change

Let's calculate $\overline{a}$

$\overline{a}$ consists of three components:

CHANGE IN X
CHANGE IN Y
CHANGE IN Z

corresponding to du, $dv = 0$, $dw = 0$

the other two do not change

$\overline{a}$:

CHANGE IN X $= dx(u,v,w) = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}\cancel{dv} + \frac{\partial x}{\partial w}\cancel{dw} = \frac{\partial x}{\partial u}du$

CHANGE IN Y $= dy(u,v,w) = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}\cancel{dv} + \frac{\partial y}{\partial w}\cancel{dw} = \frac{\partial y}{\partial u}du$

CHANGE IN Z $= dz(u,v,w) = \frac{\partial z}{\partial u}du$

So, $\overline{a} = \left(\frac{\partial x}{\partial u}du, \frac{\partial y}{\partial u}du, \frac{\partial z}{\partial u}du\right)$

Similarly, $\overline{b} = \left(\frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial v}dv, \frac{\partial z}{\partial v}dv\right)$, $\overline{c} = \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right)dw$

$$dV = \left\| \begin{matrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & \frac{\partial z}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & \frac{\partial z}{\partial v}dv \\ \frac{\partial x}{\partial w}dw & \frac{\partial y}{\partial w}dw & \frac{\partial z}{\partial w}dw \end{matrix} \right\| = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{matrix} \right\| du\,dv\,dw$$