Second partials test

1-D

$$f(x) = f(x_0) + \cancel{f'(x_0)}(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \mathcal{E}\left((x_0-x)^2\right)$$

$\cancel{f'(x_0)(x-x_0)} \to 0$

If x_0 is the point of extremuma $\Rightarrow f'(x_0)=0$ (NECESSARY condition)

Locally

$f(x) > f(x_0)$ if $f''(x_0) > 0 \Rightarrow$ local min

$f(x) < f(x_0)$ if $f''(x_0) < 0 \Rightarrow$ local max

$f(x)$? $f(x_0)$ if $f''(x_0) = 0$

2-D

$$f(x,y) = f(x_0,y_0) + \cancel{\bar{\nabla} f(x_0,y_0) \cdot \langle x-x_0, y-y_0\rangle} + \frac{1}{2}\langle x-x_0, y-y_0\rangle \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y} \end{pmatrix} \langle x-x_0, y-y_0\rangle + \mathcal{E}(..)$$

$\cancel{\bar{\nabla} f(x_0,y_0) \cdot \langle x-x_0, y-y_0\rangle} \to 0$

IF $\langle x_0,y_0\rangle$ is the point of extremuma $\Rightarrow \bar{\nabla} f(x_0,y_0) = 0$ (necessary cond.)

(f has to be differentiable)

Theorem **C.** If $f(x,y)$ has continuous second partial derivatives

INTRODUCE

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

Locally

$f(x,y) > f(x_0,y_0)$ if $D>0$ and $f_{xx}>0 \Rightarrow$ local min — #3 is positive

$f(x,y) < f(x_0,y_0)$ if $D>0$ and $f_{xx}<0 \Rightarrow$ local MAX — #3 is neg

$f(x,y)$? $f(x_0,y_0)$ if $D<0 \Rightarrow$ saddle point — #3 CAN BE BOTH

??? ? $f(x,y)$??? $f(x_0,y_0)$ if $D=0 \Rightarrow$ inconclusive