Proof of the 2-D CHAIN RULE

$g(x,y)$ is differentiable $\Leftrightarrow$

$$g(x+h_x, y+h_y) - g(x,y) = \frac{\partial g}{\partial x} h_x + \frac{\partial g}{\partial y} h_y + O(h_x, h_y)$$

Consider three differentiable functions

$$f(x,y),\ s(x,y),\ t(x,y)$$

WHAT ABOUT DIFFERENTIABILITY of

$$f(s(x,y), t(x,y)) - ?$$

CHECK:

$$f(s(x+h_x, y+h_y), t(x+h_x, y+h_y)) - f(s(x,y), t(x,y)) =$$

$$= \frac{\partial f}{\partial x}\left[s(x+h_x, y+h_y) - s(x,y)\right] + \frac{\partial f}{\partial y}\left[t(x+h_x, y+h_y) - t(x,y)\right] + O\left(\begin{smallmatrix}\text{Blah}\\ \text{Blah}\end{smallmatrix}\right)$$

$$= \frac{\partial f}{\partial x}\left[\frac{\partial s}{\partial x} h_x + \frac{\partial s}{\partial y} h_y + O(\text{Blah})\right] + \frac{\partial f}{\partial y}\left[\frac{\partial t}{\partial x} h_x + \frac{\partial t}{\partial y} h_y + O(\text{Blah})\right] + O(\text{Blah}, \text{Blah})$$

$$= \left\{\frac{\partial f}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial t}{\partial x}\right\} h_x + \left\{\frac{\partial f}{\partial x}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial t}{\partial y}\right\} h_y + O(\text{Blah}, \text{Blah}, \text{Blah})$$

$$\left\{\frac{\partial f}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial t}{\partial x}\right\} = \frac{\partial}{\partial x}\left[f(s(x,y), t(x,y))\right] \qquad \left\{\frac{\partial f}{\partial x}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial t}{\partial y}\right\} = \frac{\partial}{\partial y}\left[f(s(x,y), t(x,y))\right]$$