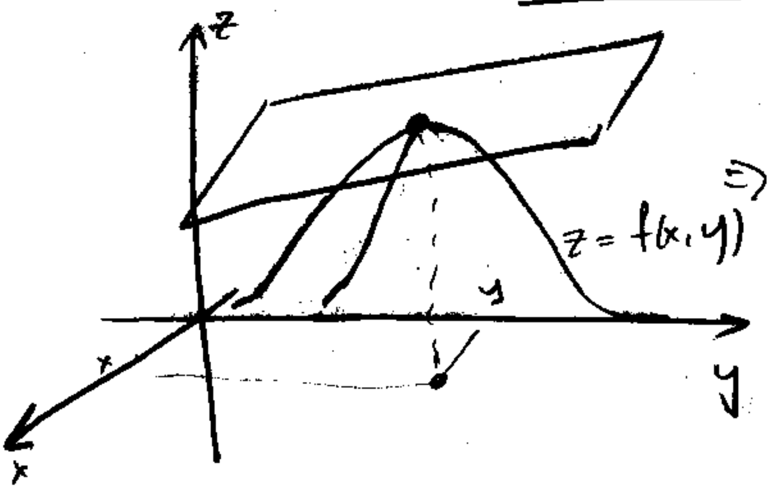
Differentiability in 2-D

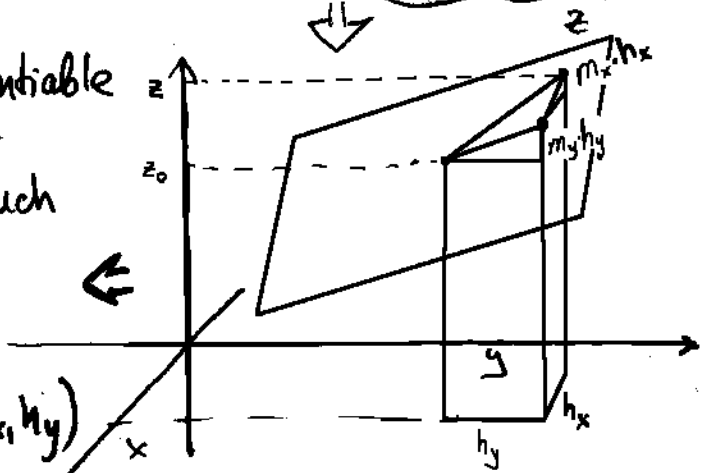


A 2-D function $z = f(x,y)$ CAN BE approximated by a plane

Later: the plane of the best approximation will be taken by definition for the tangent plane

Def: A function $f(x,y)$ is differentiable at (x,y) if there exist numbers m_x and m_y such that

$$\otimes \quad f(x+h_x, y+h_y) = f(x,y) + \underbrace{m_x h_x + m_y h_y}_{\text{Differential}} + o(h_x, h_y)$$



motion in plane

$$z = z_0 + m_x h_x + m_y h_y$$

$o(h_x, h_y)$ is the quantity of the order of magnitude smaller than (h_x, h_y)

Find m_x:

Fix $h_y \equiv 0 \Rightarrow$

$$\frac{f(x+h_x, y) - f(x,y)}{h_x} = m_x + \frac{o(h_x, h_y)}{h_x} \to 0$$

as $h_x \to 0$

$$\frac{\partial f}{\partial x}(x,y) = m_x, \quad \text{also} \quad m_y = \frac{\partial f}{\partial y}(x,y)$$

APPROX. of A function

$$f(x+h_x, y+h_y) \cong f(x,y) + \frac{\partial f}{\partial x} h_x + \frac{\partial f}{\partial y} h_y$$

Best approximation is by the tangent plane $\Rightarrow$

$\Rightarrow$ tangent plane is

$$z = f(x,y) + \frac{\partial f}{\partial x} h_x + \frac{\partial f}{\partial y} h_y,$$

or,

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

or

$$z = f(x_0,y_0) + \nabla f \cdot ((x-x_0),(y-y_0))$$

Differential

Gradient:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

$$df(x,y) = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$