Fundamental Theorem of Calculus. Part I:

Connection between integration and differentiation

Motivation:

Problem of finding antiderivatives

Definition: An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

In other words, given the function $f(x)$, you want to tell whose derivative it is.

Example 1. Find an antiderivative of 1.

An answer: x.

Example 2. Find an antiderivative of $\frac{1}{1+x^2}$.

An answer: $\arctan x$.

!!..??!..!?

HOW DO YOU KNOW?

Some antiderivatives can be found by reading differentiation formulas backwards:

$$[x^\alpha]' = \alpha x^{\alpha-1}$$

$$[\cos x]' = -\sin x$$

$$[\tan x]' = \frac{1}{\cos^2 x}$$

$$\boldsymbol{[\arctan x]' = \frac{1}{1+x^2}}$$

$$[\arcsin x]' = \frac{1}{\sqrt{1-x^2}}$$

Limitation: there aren't formulas for

$$\sin(x^2), \qquad e^{(x^2)},$$

$$\sqrt{x + \sin^2(1 - x^3)}$$

?!?

Do these functions have antiderivatives?

An observation

No matter what object's velocity $v(t)$ **is, its position** $d(t)$ **is always an antiderivative of** $v(t)$: $d'(t) = v(t)$. **This suggests that all functions have antiderivatives.**

Antiderivative of $\sin(t^2)$ exists!

Suppose the speed of my car obeys $\sin(t^2)$ (do not try it on the road!). The car will move accordingly and the position of the car $F(t)$ will give the antiderivative of $\sin(t^2)$.

A hypothesis

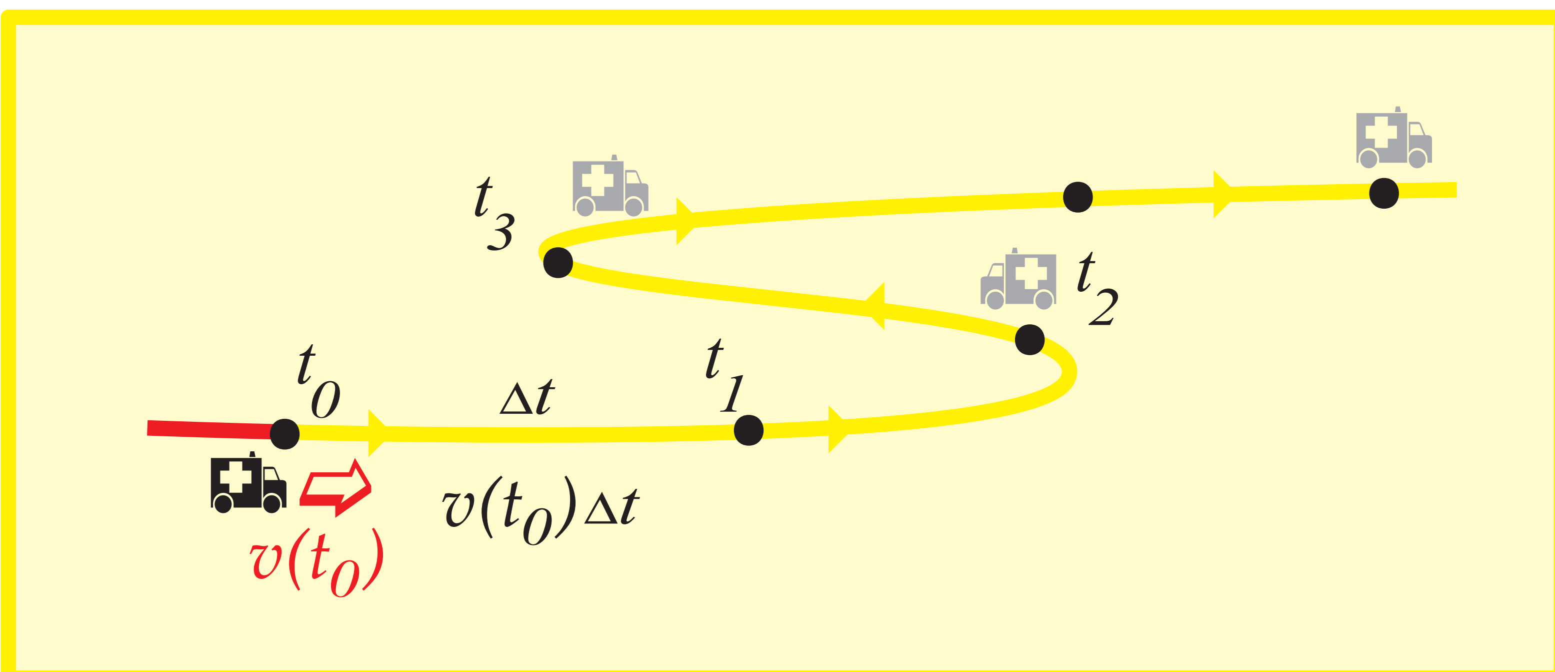
Calculating antiderivatives must be similar to calculating position from velocity

Fundamental Theorem of Calculus

Naive derivation

Let, at initial time t_0, position of the car on the road is $d(t_0)$ and velocity is $v(t_0)$. After a short period of time Δt, the new position of the car is approximately

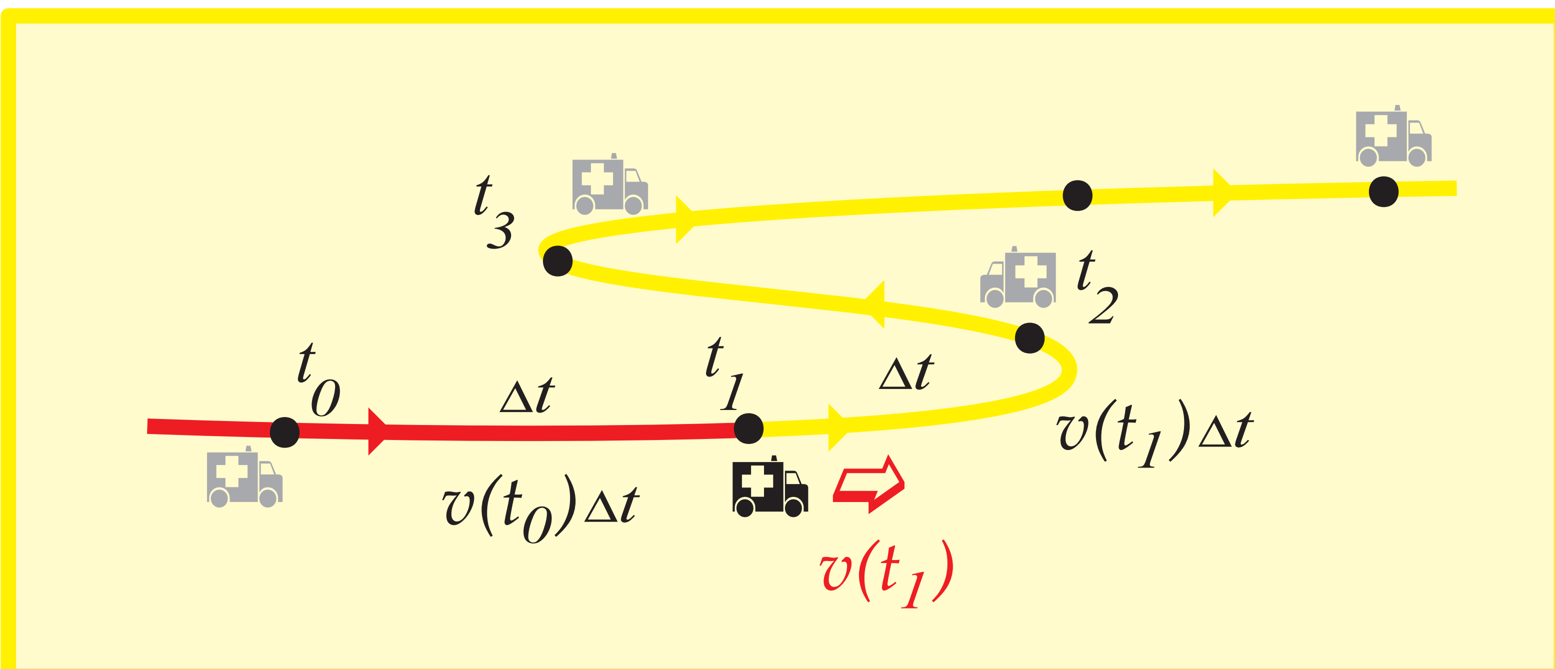
$$d(t_1) \approx d(t_0) + v(t_0)\Delta t, \quad (t_1 = t_0 + \Delta t)$$



After two moments of time

$$d(t_2) \approx d(t_0) + v(t_0)\Delta t + v(t_1)\Delta t,$$

$$(t_2 = t_1 + \Delta t)$$

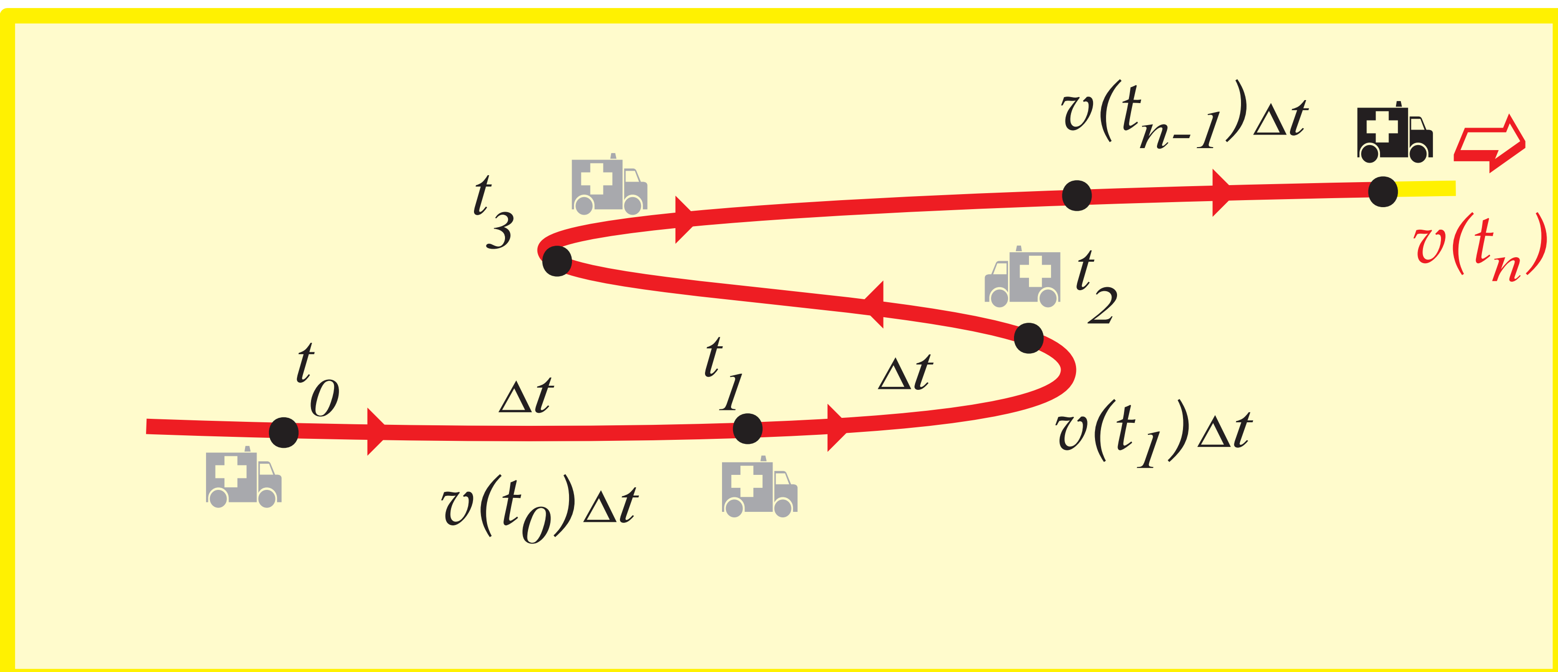


After n moments of time

$$d(t_n) \approx d(t_0) + v(t_0)\Delta t + \ldots + v(t_{n-1})\Delta t$$

$$= d(t_0) + \sum_{i=0}^{n-1} v(t_i)\Delta t$$

$$(t_n = t_0 + n\Delta t)$$



As $\Delta t \to 0$, $(\Delta t = (t - t_0)/n)$.

$$d(t) - d(t_0) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} v(t_i)\Delta t$$

Compare this to

$$\int_{t_0}^{t} v(\tau)\mathrm{d}\tau = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} v(t_i)\Delta t$$

Combining the two formulas

$$d(t) - d(t_0) = \int_{t_0}^{t} v(\tau)\mathrm{d}\tau$$

while

$$d'(t) = v(t)$$

Assuming $t_0 = 0$, $d(t_0) = 0$, distance can be calculated from velocity by

$$d(t) = \int_0^t v(\tau)\mathrm{d}\tau$$

Is this a function in our regular sense? Yes, for each value of t it defines a unique number.

If $d'(t) = v(t)$? Yes. This constitutes the assertion of Fundamental Theorem of Calculus.

THEOREM. Let $f(x)$ be a continuous function (so, the definite integral of $f(x)$ exists). Then the function

$$F(x) = \int_a^x f(\tau)\mathrm{d}\tau.$$

is an antiderivative of $f(x)$, $F'(x) = f(x)$.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

EXAMPLES

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 3. **Find an antiderivative of**

$$f(x) = \sin(x^2).$$

An answer: $F(x) = \int_0^x \sin(\tau^2)\mathrm{d}\tau.$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\,\mathrm{d}\tau\right] = f(x)}$$

Example 4. **Find an antiderivative of**

$$f(x) = \mathrm{e}^{(x^2)}.$$

An answer: $G(x) = \int_0^x \mathrm{e}^{(\tau^2)}\mathrm{d}\tau.$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 5. **Find an antiderivative of**

$$f(x) = \sqrt{x + \sin^2(1 - x^3)}.$$

An answer:

$$H(x) = \int_0^x \sqrt{\tau + \sin^2(1 - \tau^3)}\mathrm{d}\tau.$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 6. **Calculate $F'(x)$ if**

$$F(x) = \int_0^x \sin(\tau^2)\mathrm{d}\tau$$

Answer: $\sin(x^2)$.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 7. **Calculate $G'(x)$ if**

$$G(x) = \int_0^x \mathrm{e}^{(\tau^2)}\mathrm{d}\tau$$

Answer: $\mathrm{e}^{(x^2)}$.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 8. **Calculate $H'(x)$ if**

$$H(x) = \int_0^x \sqrt{\tau + \sin^2(1 - \tau^3)}\mathrm{d}\tau$$

Answer: $\sqrt{x + \sin^2(1 - x^3)}$.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)}$$

Example 9. **Calculate $F'(x)$ if**

$$F(x) = \int_x^0 \sin(\tau^2)\mathrm{d}\tau$$

Solution: $$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_x^0 \sin(\tau^2)\mathrm{d}\tau\right] =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[-\int_0^x \sin(\tau^2)\mathrm{d}\tau\right] = -\sin(x^2)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)$$

Example 10. Calculate $G'(x)$ if

$$G(x) = \int_1^{5x} \sin(\tau^2)\mathrm{d}\tau$$

Trick. Introduce $F(x) = \int_1^x \sin(\tau^2)\mathrm{d}\tau$, so, $G(x) = F(5x)$.

Use FTC to calculate $F'(x) = \sin(x^2)$.

Use the Chain Rule to find the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}G(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(5x)$$

$$= F'(5x)[5x]' = \sin((5x)^2)5.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(\tau)\mathrm{d}\tau\right] = f(x)$$

Example 11. Calculate $H'(x)$ if

$$H(x) = \int_{\pi}^{x^3} \sin(\tau^2)\mathrm{d}\tau$$

Trick. Introduce $F(x) = \int_{\pi}^{x} \sin(\tau^2)\mathrm{d}\tau$, so, $H(x) = F(x^3)$.

Use FTC to calculate $F'(x) = \sin(x^2)$.

Use the Chain Rule to find the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}H(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x^3)$$

$$= F'(x^3)[x^3]' = \sin((x^3)^2)3x^2.$$

Example 12. Calculate $G'(x)$ if

$$G(x) = \int_{\sqrt{x}}^{x^3} \sin(\tau^2)\mathrm{d}\tau$$

Solution: First, notice that

$$G(x) = \int_{\sqrt{x}}^{0} \sin(\tau^2)\mathrm{d}\tau + \int_{0}^{x^3} \sin(\tau^2)\mathrm{d}\tau$$

Trick. Introduce $F(x) = \int_0^x \sin(\tau^2)\mathrm{d}\tau$, so,

$$G(x) = -F(\sqrt{x}) + F(x^3).$$

Use FTC to calculate $F'(x) = \sin(x^2)$.

Use the Chain Rule to find the derivative

$$\frac{d}{dx}G(x) = -\frac{d}{dx}F(\sqrt{x}) + \frac{d}{dx}F(x^3)$$

$$= -F'(\sqrt{x})[\sqrt{x}]' + F'(x^3)[x^3]'$$

$$= -\sin((\sqrt{x})^2)\frac{1}{2\sqrt{x}} + \sin((x^3)^2)3x^2.$$

Control exercises

Example 13. Calculate $F'(x)$ if

$$F(x) = \int_{1-x^2}^{1+x^2} \sin(\tau^2)\mathrm{d}\tau$$

Example 14. Calculate $G'(x)$ if

$$G(x) = \int_{1}^{x} x^2 \sin(\tau^2)\mathrm{d}\tau$$

Fundamental Theorem of Calculus

Proof of Part I

We will be using two facts

Interval Additive Property:

$$\int_a^b f(\tau)\mathrm{d}\tau = \int_a^c f(\tau)\mathrm{d}\tau + \int_c^b f(\tau)\mathrm{d}\tau$$

Comparison Property: **If** $m \leq f(x) \leq M$ **on** $[a, b]$

$$m(b-a) \leq \int_a^b f(\tau)\mathrm{d}\tau \leq M(b-a)$$

Let $F(x) = \int_a^x f(\tau)\mathrm{d}\tau$.

We will now show that $F'(x) = f(x)$.

By definition

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(\tau)\mathrm{d}\tau - \int_a^x f(\tau)\mathrm{d}\tau}{h}$$

Notice, that by the Interval Additive Property

$$\int_a^{x+h} f(\tau)\mathrm{d}\tau - \int_a^{x} f(\tau)\mathrm{d}\tau = \int_x^{x+h} f(\tau)\mathrm{d}\tau$$

Therefore,

$$F'(x) = \lim_{h\to 0} \frac{\int_a^{x+h} f(\tau)\mathrm{d}\tau - \int_a^{x} f(\tau)\mathrm{d}\tau}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(\tau)\mathrm{d}\tau$$

Also, by the Comparison Property, for

$$m = \min_{\tau \in [x, x+h]} f(\tau), \qquad M = \max_{\tau \in [x, x+h]} f(\tau)$$

one has

$$mh \le \int_x^{x+h} f(\tau) \mathrm{d}\tau \le Mh$$

or, dividing by h,

$$m \le \frac{1}{h} \int_x^{x+h} f(\tau) \mathrm{d}\tau \le M$$

Finally, by the Squeeze Theorem, expression

$$m \le \frac{1}{h}\int_x^{x+h} f(\tau)\mathrm{d}\tau \le M$$

converges to

$$f(x) \le \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(\tau)\mathrm{d}\tau \le f(x)$$

since both $m = \min f(\tau)$ **and** $M = \max f(\tau)$ **on** $[x, x+h]$ **coincide with** $f(x)$ **as** $h \to 0$

The theorem is proved.