Limit of a function at a point

$\varepsilon-\delta$ language
Motivation:

Studying functions when they are not defined
The following functions are undefined at $x = 1$:

\[ f(x) = \frac{x^2 + 1}{x - 1} \]

\[ f(x) = \frac{x^2 - 1}{x - 1} \]
The following functions are undefined at $x = 1$:

$$f(x) = \frac{x^2 + 1}{x - 1}$$

$$f(x) = \frac{x^2 - 1}{x - 1}$$

The difference can be big!
Want to distinguish the following situation:
Want to distinguish the following situation:

As $x$ is near 1, value of $f(x) = \frac{x^2 - 1}{x - 1}$ is near 2.
HOW EXACTLY NEAR?
HOW EXACTLY NEAR?

This near?

0.9 1 1.1
HOW EXACTLY NEAR?

This near?

This near?
HOW EXACTLY NEAR?

This near?

This near?

Or, this near?
!!...!!!!...!!

INFINITELY NEAR!
INFINITELY NEAR?
??...????...??

INFINITELY NEAR?
\( \varepsilon - \delta \) language.

Definition of Limit

Working out the infinity
Definition of Limit

DEFINITION. The number $L$ is the limit of function $f(x)$ as $x$ approaches $c$ if and only if for any positive number $\varepsilon$ there exists a positive number $\delta$ (depending on $\varepsilon$) such that as long as $x$ is not equal to $c$ but differs from $c$ by less than $\delta$, it implies that $f(x)$ differs from $L$ by less than $\varepsilon$. 
Limit in math symbols.

DEFINITION.

\[ L = \lim_{x \to c} f(x) \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \]

Legend: \( \forall \) — for any, \( \varepsilon \) — “epsilon”, \( \exists \) — exists, \( \delta \) — “delta”, \( / \) — such that, \( \Rightarrow \) — implies, \( \iff \) — if and only if, \( \to \) — approaches.
\[ L = \lim_{x \to c} f(x) \quad \iff \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \]

The inequality in red requires that

\[-\delta < x - c < \delta, \quad x - c \neq 0\]

or,

\[c - \delta < x < c + \delta, \quad x \neq c\]
\[
L = \lim_{x \to c} f(x) \iff \forall \varepsilon > 0 \exists \delta > 0 / 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

The inequality in green requires that

\[
-L + \varepsilon < f(x) < L - \varepsilon,
\]

or,

\[
-L - \varepsilon < f(x) < L + \varepsilon,
\]
\[
L = \lim_{x \to c} f(x) \iff \\
\forall \varepsilon > 0 \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

As \( x \) is in \( \delta \)-corridor, \( f(x) \) is in \( \varepsilon \)-corridor:

(A narrower \( \delta \)-corridor guarantees it better)
\[ L = \lim_{x \to c} f(x) \iff \forall \varepsilon > 0 \exists \delta > 0 / 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \]

For every choice of \( \varepsilon \) there must exist \( \delta \):

(it is highly desirable to have a formula for computing \( \delta \) from \( \varepsilon \))
\[ L = \lim_{x \to c} f(x) \iff \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \]

No limit situations:
EXAMPLES
\[ L = \lim_{x \to c} f(x) \quad \iff \quad \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \]

**EXAMPLE 1.** Prove by \( \varepsilon-\delta \) argument

\[ \lim_{x \to 2} (7x + 1) = 15 \quad \iff \]
\[
L = \lim_{x \to c} f(x) \iff \\
\forall \varepsilon > 0 \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

**EXAMPLE 1.** Prove by \(\varepsilon-\delta\) argument

\[
\lim_{x \to 2} (7x + 1) = 15 \iff \\
\forall \varepsilon > 0 \exists \delta > 0 \ / \ 0 < |x - 2| < \delta \Rightarrow |(7x + 1) - 15| < \varepsilon
\]
\[ L = \lim_{x \to c} f(x) \iff \]

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \ \Rightarrow \ |f(x) - L| < \varepsilon
\]

**EXAMPLE 1.** Prove by \(\varepsilon-\delta\) argument

\[
\lim_{x \to 2}(7x + 1) = 15 \iff
\]

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - 2| < \delta \ \Rightarrow \ |(7x+1) - 15| < \varepsilon
\]

(By a smart choice of \(\delta\) guarantee that

\[
|(7x + 1) - 15| < \varepsilon
\]
The desired inequality

\[|(7x + 1) - 15|\]
The desired inequality

\[ |(7x + 1) - 15| = |7x - 14| = |7(x - 2)| \]

\[ = |7||x - 2| = 7|x - 2| < \varepsilon \quad \text{(desirable)} \]
The desired inequality

\[ |(7x + 1) - 15| = |7x - 14| = |7(x - 2)| \]

\[ = |7||x - 2| = 7|x - 2| < \varepsilon \quad \text{(desirable)} \]

follows from the assumption

\[ |x - 2| < \delta \]

if \( \delta \leq \varepsilon / 7 \).
The desired inequality

\[
|(7x + 1) - 15| = |7x - 14| = |7(x - 2)|
\]

\[
= |7||x - 2| = 7|x - 2| < \varepsilon \quad \text{(desirable)}
\]

follows from the assumption

\[
|x - 2| < \delta
\]

if

\[
\delta \leq \varepsilon/7.
\]

In particular, one can pick

\[
\delta = \varepsilon/7. \quad \text{(answer)}
\]
\[ L = \lim_{x \to c} f(x) \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \]
\[ L = \lim_{x \to c} f(x) \quad \iff \quad \forall \varepsilon > 0 \, \exists \delta > 0 \, / \, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \]

**EXAMPLE 2.** Prove by \( \varepsilon-\delta \) argument

\[
\lim_{x \to 5} \left( \frac{x^2 - 25}{x - 5} \right) = 10 \quad \iff
\]
\[
L = \lim_{x \to c} f(x) \implies \\
\forall \varepsilon > 0 \ \exists \delta > 0 / \ 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon
\]

**EXAMPLE 2.** Prove by \(\varepsilon-\delta\) argument

\[
\lim_{x \to 5} \left( \frac{x^2 - 25}{x - 5} \right) = 10 \implies \\
\forall \varepsilon > 0 \ \exists \delta > 0 / \ 0 < |x - 5| < \delta \implies \left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon
\]

(By a smart choice of \(\delta\) guarantee that

\[
\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon
\] )
The desired inequality

$$\left|\frac{x^2 - 25}{x - 5} - 10\right|$$
The desired inequality

\[
\left| \frac{x^2 - 25}{x - 5} - 10 \right| = \left| \frac{(x + 5)(x - 5)}{x - 5} - 10 \right|
\]

\[
= |x - 5| < \varepsilon \quad \text{(desirable)}
\]

follows from the assumption

\[
|x - 5| < \delta
\]

if

\[
\delta \leq \varepsilon.
\]

In particular, one can pick

\[
\delta = \varepsilon. \quad \text{(answer)}
\]
\[ L = \lim_{x \to c} f(x) \quad \iff \quad \forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \]

EXAMPLE 3. Prove by \( \varepsilon - \delta \) argument

\[ \lim_{x \to 3} x^2 = 9 \quad \iff \]
\[
L = \lim_{x \to c} f(x) \iff \\
\forall \varepsilon > 0 \ \exists \delta > 0 / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

**EXAMPLE 3.** Prove by \( \varepsilon-\delta \) argument

\[
\lim_{x \to 3} x^2 = 9 \iff \\
\forall \varepsilon > 0 \ \exists \delta > 0 / \ 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon
\]

(By a smart choice of \( \delta \) guarantee that

\[
|x^2 - 9| < \varepsilon
\]  )
The desired inequality

$$|x^2 - 9|$$
The desired inequality

\[ |x^2 - 9| = |(x - 3)(x + 3)| \]

\[ = |x - 3||x + 3| < \varepsilon \quad \text{(desirable)} \]
The desired inequality

\[ |x^2 - 9| = |(x - 3)(x + 3)| \]

\[ = |x - 3||x + 3| < \varepsilon \quad \text{(desirable)} \]
requires controlling both \( |x - 3| \) and \( |x + 3| \).

Note that \( \delta \) controls \( |x - 3| \) through

\[ |x - 3| < \delta \]

Does \( \delta \) controls \( |x + 3| \) as well?
Does $\delta$ controls $|x + 3|$ as well?

Assume $\delta < 1$: 
Does $\delta$ controls $|x + 3|$ as well?

Assume $\delta < 1$:

$$|x - 3| < 1 \iff 2 < x < 4$$
Does $\delta$ controls $|x + 3|$ as well?

Assume $\delta < 1$:

$$|x - 3| < 1 \iff 2 < x < 4$$

Notice that if $2 < x < 4$, then

$$5 < |x + 3| < 7 \quad (\delta \text{ controls } |x + 3|!)$$
Does \( \delta \) controls \(|x + 3|\) as well?

Assume \( \delta < 1 \):

\[
|x - 3| < 1 \iff 2 < x < 4
\]

Notice that if \( 2 < x < 4 \), then

\[
5 < |x + 3| < 7 \quad (\delta \text{ controls } |x + 3|!)
\]

Finally, \(|x^2 - 9| = |x - 3||x + 3| < |x - 3|7 < \varepsilon \) if \( \delta < 1 \) and \( \delta \leq \varepsilon/7 \).

Answer: \( \delta = \min\{1, \varepsilon/7\} \)