Well-Posed Initial-Boundary Value Constrained Evolution Problems

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Outline

- A constrained evolution system;
- Constraint preserving boundary conditions;
- The static constraint evolution reduction;
- Well-posed IBVP for the constraint evolution;
- Well-posedness of free-evolution problem.

A constrained evolution system

($\Delta u_i = \sum_{l=1}^{3} \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l} u_i = \partial^l \partial_l u_i$, div$u = \partial^i u_i$)

Vector wave equation, $u_i(x) : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$\partial_t^2 u_i = \partial^l \partial_l u_i \qquad \text{(Evolution equations)}$$

subject to the divergence-free constraint

$$\partial^i u_i = 0. \qquad \text{(Constraint equations)}$$

Need to find a solution in a bounded region Ω.

(Maxwell's Equations, Navier-Stokes equations, Einstein's equations)

Static constraint evolution system

Constrained evolution

$$\begin{cases} \partial_t^2 u_i = \partial^l \partial_l u_i \\ C := \partial^i u_i = 0 \end{cases}$$

Static constraint evolution system

Constrained evolution

$$\begin{cases} \partial_t^2 u_i = \partial^l \partial_l u_i \\ C := \partial^i u_i = 0 \end{cases} \qquad \Rightarrow \qquad \boxed{\partial_t^2 C = \partial^l \partial_l C}$$

Static constraint evolution system

Constrained evolution

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$\Rightarrow$

$$\boxed{\partial_t^2 C = \partial^l \partial_l C}$$

Free evolution

$$\partial_t^2 u_i = \partial^l \partial_l u_i$$

BCs: $2 + 1$ on $\partial\Omega$

$$C = 0$$

$$\frac{\partial}{\partial n} C = 0$$

$$\partial_t C + \frac{\partial}{\partial n} C = 0$$

Static constraint evolution system

Constrained evolution

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$$\partial_t C + \frac{\partial}{\partial n} C = 0$$

$$\Downarrow \quad \Uparrow$$

Constrained evolution #2

$$\begin{cases} \partial_t^2 u_i = \partial^l \partial_l u_i - \partial_i C \\ \partial^i u_i = 0 \end{cases}$$

$$\boxed{\partial_t^2 C = 0}$$

Static constraint evolution system

Constrained evolution

$$\begin{cases} \partial_t^2 u_i = \partial^l \partial_l u_i \\ C := \partial^i u_i = 0 \end{cases}$$

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Free evolution

$$\partial_t^2 u_i = \partial^l \partial_l u_i$$

BCs: $2 + 1$ on $\partial\Omega$

$C = 0$

$\frac{\partial}{\partial n} C = 0$

$\partial_t C + \frac{\partial}{\partial n} C = 0$

BCs: (natural?)

$\partial_t u_A + \partial_n u_A - \partial_A u_n = 0$

$\partial_t u_n + \partial_n u_A = 0$

$$\partial_t^2 u = \mathbf{curl}\,\mathbf{curl}\, u$$

Static constraint evolution reduction

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j$$

$$\partial^i u_i = 0 \qquad (\partial_t^2 \partial^i u_i = 0)$$

Static constraint evolution reduction

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j$$
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Recall: $(\mathbf{curl}\, u)_k = \varepsilon_k^{\ lm} \partial_l u_m$, also $\varepsilon_{ij}^{\ \ k} \varepsilon_k^{\ lm} = \delta_i^l \delta_j^m - \delta_j^l \delta_i^m$

Static constraint evolution reduction

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Finally, $\partial^j \partial_j u_i - \partial_i \partial^j u_j = \partial^j [\delta_i^l \delta_j^m - \delta_j^l \delta_i^m] \partial_l u_m = \partial^j \varepsilon_{ij}^k \varepsilon_k^{lm} \partial_l u_m$
$= \mathbf{curl} \mathbf{curl} u$

Static constraint evolution reduction

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$= \mathbf{curlcurl} u$

First order decomposition: define $v = \partial_t u$, $w = \mathbf{curl} u$

$$\partial_t v = \mathbf{curl} u$$
$$\partial_t w = \mathbf{curl} v$$

specify BCs, use the solution for second order system.

Well-posed IVBP for constraint evolution

(1) $\partial_t^2 u_i = \partial^j \partial_j u_i$

(2) $\partial^i u_i = 0$

Theorem 1. The constrained evolution problem (1),(2) has a unique solution satisfying boundary conditions

$$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha_A^B (\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A, \ \|\alpha\| \leq 1$$

provided that the initial data satisfies the constraint (2). In addition, the solution satisfies the energy estimate

$$\sup_{0 \leq t \leq T} [\|\partial_t u_i\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}\|_{L^2(\Omega)}]$$
$$\leq c \int_0^T \|g\|_{L^2(\partial\Omega)} dt + [\|\partial_t u_i(0)\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}(0)\|_{L^2(\Omega)}].$$

The free-evolution problem

(1) $\partial_t^2 u_i = \partial^j \partial_j u_i$

Theorem 2. The free evolution problem (1) has a unique solution satisfying boundary conditions

(3) $$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A,\ |\alpha| \leq 1$$
$$(\partial_t u_n + \partial_n u_n) = -\alpha(\partial_t u_n - \partial_n u_n) + g$$

provided that the initial data satisfies the constraint (2) and functions g and g_A satisfy the compatibility conditions $\partial_t g = -\partial^A g_A,\ g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0)$, on $\partial\Omega$.

The solution satisfies constraint (2) and the energy estimate of Theorem 1.

The Laplace-Fourier analysis

Consider first-order reduction of (1): ($\partial_t u_i = \pi_i$, $\varphi_{ki} = \partial_k u_i$)

$$\partial_t \pi_i = \partial^j \varphi_{ji}$$

$$\partial_t \varphi_{ji} = \partial_j \pi_i$$

The Laplace-Fourier analysis

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$$\partial_t \pi_i = \partial^j \varphi_{ji}$$
$$\partial_t \varphi_{ji} = \partial_j \pi_i$$

We seek a solution as a superposition of modes

$$\begin{pmatrix} \pi_i \\ \varphi_{ji} \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_i(x_1) \\ \tilde{\varphi}_{ji}(x_1) \end{pmatrix} \exp(st + \imath \omega_A x^A),$$

where $s \in \mathbb{R}$, $\Re(s) > 0$, ω_A is a real two-vector

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Kreiss condition:

$$\|(\tilde{\pi}_i(0), \tilde{\varphi}_{xi}(0))\| \leq K\|g, g_A\| \quad (\leq K\|h, h_A\|, \quad \leq K\|f_A\|)$$

The Kreiss condition is satisfied for (3)

The proposed result

(1) $\partial_t^2 u_i = \partial^j \partial_j u_i$

Theorem 3. (Constraint (2) is not enforced) Free evolution problem (1) has a unique solution satisfying

(3) $$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A,\ |\alpha| \leq 1$$
$$(\partial_t u_n + \partial_n u_n) = -\alpha(\partial_t u_n - \partial_n u_n) + g$$

provided that functions g and g_A satisfy $\partial_t g = -\partial^A g_A$, $g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0)$, on $\partial\Omega$.
In addition, the solution satisfies the energy estimate

$$\sup_{0\leq t\leq T} [\|\partial_t u_i\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}\|_{L^2(\Omega)} + \|\partial^i u_i\|_{L_2(\Omega)}] \leq \text{I+B data}$$

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