Well-Posed Initial-Boundary Value Problem for Constrained Evolution

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Design of parallel algorithms for Einstein's equations requires a better understanding of conditions that one can specify at the artificial boundary. The special interest is paid to conditions capable of trading "physical" radiation through the boundary.

We propose a new way of constructing boundary conditions and a new way of specifying IBVPs for Einstein Equations using static constrain evolution reductions.

Below we give two model problem of constrained evolution. For the first (constrained) problem, a well-posed IBVP is constructed, and a well-posed free evolution problem can be proposed based on the new approach. For the second problem, more investigation is needed to establish existence. Trivial constraint evolution reductions and stability results, however, can are obtained.

1. Vector wave equation: 3-2=1

Consider a vector wave equation in a polyhedral domain Ω,

$$\partial_t^2 u_i = \partial^j \partial_j u_i, \tag{1}$$

(indices are raised and lowered using the flat metric) subject to the divergence-free constraint

$$C := \partial^i u_i = 0. \tag{2}$$

It can be easily verified that the constraint quantity C obeys a subordinate wave equation

$$\partial_t^2 C = \partial^i \partial_i C. \tag{3}$$

Thus, if $C \neq 0$ at some point of the boundary, in view of (3), the constraint perturbation will flow inside the domain. In general to ensure $C \equiv 0$ the boundary data must be given consistently with the evolution of the constraint quantity. The data that guarantees that C remains zero for all times is called *constraint-preserving*.

Many approaches have been introduced to enforce compatibility of boundary data. Making an analogy to problem (1), (2), the conditions enforcing either of

$$C|_{\partial\Omega} = 0, \quad \partial_n C|_{\partial\Omega} = 0, \quad \text{or}$$

$$(\partial_t C + \partial_n C)|_{\partial\Omega} = 0 \tag{4}$$

would have been used in the majority of the approaches. For example, an elaborate integration along the boundary is employed to produce a compatible boundary data for the problem. In other cases, boundary conditions on certain incoming modes were simply replaced with the above conditions, and Laplace-Fourier techniques used to prove well-posedness of the resulting problem. It is, however, not completely clear how to justify selection of the modes that will be replaced with (4). Also, in the first case, it is not clear if all solutions of the original constraint system can be obtained this way.

We propose to replace (1), (2) with a different constrained system. Namely, we consider equation

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j \tag{5}$$

subject to constraint (2).

It is straightforward to verify that (5) implies the subordinate equation

$$\partial_t^2 C = 0. \tag{6}$$

This means that equation (5) also preserves the constraint, but in contrast to (1), in the new system, constraint preservation depends exclusively on the initial data.

The following result can be formulated [1]:

Theorem 1. The constrained evolution problem (5), (2) has a unique solution satisfying boundary conditions

$$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha_A^B(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A, \quad \|\alpha\| \leq 1 \tag{7}$$

provided that the initial data satisfies the constraint (2). In addition, the solution satisfies the energy estimate

$$\sup_{0\leq t\leq T} [\|\partial_t u_i\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}\|_{L^2(\Omega)}]$$
$$\leq c\int_0^T \|g\|_{L^2(\partial\Omega)} dt + [\|\partial_t u_i(0)\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}(0)\|_{L^2(\Omega)}].$$

It is often advantageous from the numerical point of view to solve the free evolution problem versus the fully constrained problem. We recall that only the evolution equation (1) is solved in the free evolution problem, while constraint (2) is monitored but not actively enforced on the solution. The following theorem gives an example of radiation-controlling boundary data that guarantee preservation of the constraint.

Theorem 2. The free evolution problem (1) has a unique solution satisfying boundary conditions

$$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A, \quad |\alpha| \leq 1 \tag{8}$$

$$(\partial_t u_n + \partial_n u_n) = -\alpha(\partial_t u_n - \partial_n u_n) + g, \qquad (9)$$

provided that the initial data satisfies the constraint (2) and functions g and g_A satisfy the compatibility conditions

$$\partial_t g = -\partial^A g_A, \quad g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0) \quad \text{on} \quad \partial\Omega. \qquad (10)$$

In addition, the solution satisfies constraint (2) and the energy estimate of Theorem 1.

It is not practical, however, to assume that (2) is satisfied exactly by the initial data, therefore it is proposed to strengthen Theorem 2 with the following result.

Theorem 3. (proposed) Free evolution problem (1) has a unique solution satisfying

$$(\partial_t u_A + 2\partial_{[n} u_{A]}) = \alpha(\partial_t u_B - 2\partial_{[n} u_{B]}) + g_A, \quad |\alpha| \leq 1$$

$$(\partial_t u_n + \partial_n u_n) = -\alpha(\partial_t u_n - \partial_n u_n) + g,$$

Provided that functions g and g_A satisfy $\partial_t g = -\partial^A g_A$, $g(0) = (1 + \alpha)\partial_t u_n(0) + (1 - \alpha)\partial_n u_n(0)$, on $\partial\Omega$. In addition, the solution satisfies the energy estimate (proposed)

$$\sup_{0 \leq t \leq T} [\|\partial_t u_i\|_{L^2(\Omega)} + 2\|\partial_{[j} u_{i]}\|_{L^2(\Omega)} + \|\partial^i u_i\|_{L_2(\Omega)}] \leq \text{I+B data} \quad (11)$$

The difficulty with establishing is that conditions (8), (9) are not in the maximal dissipative form for (1) (they, however, are for (5). In [1] Laplace-Fourier analysis is used to study well-posedness of (8), (9). Namely, (1) is reduced to first order in the standard way:

$$\begin{aligned} \partial_t u_i &= \pi_i, \qquad \varphi_{ki} = \partial_k u_i \\ \partial_t \pi_i &= \partial^j \varphi_{ji} \\ \partial_t \varphi_{ji} &= \partial_j \pi_i \end{aligned}$$

We seek a solution as a superposition of modes

$$\begin{pmatrix} \pi_i \\ \varphi_{ji} \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_i(x_1) \\ \tilde{\varphi}_{ji}(x_1) \end{pmatrix} \exp(st + \imath\omega_A x^A),$$

where $s \in \mathbb{C}$, $\Re(s) > 0$, ω_A is a real two-vector.

The Kreiss condition is satisfied if fields $\tilde{\pi}_i(0)$, $\tilde{\varphi}_{xi}(0)$ can be bounded by the data. It is shown in [1] that the Kreiss condition is satisfied for (8), (9), namely

$$\|(\tilde{\pi}_i(0), \tilde{\varphi}_{xi}(0))\| \leq K\|g, g_A\|$$

It is shown in [1] that the Kreiss condition is satisfied for (8), (9).

2. Six coupled wave equations: 6-3=5?

Let κ_{ij} be a symmetric matrix field in Ω, consider the system of equations

$$\partial_t^2 \kappa_{ij} = \partial^l \partial_l \kappa_{ij} \tag{12}$$

coupled through the constraint

$$M_j = \partial^i \kappa_{ij} - \partial_j \kappa^i_i = 0. \tag{13}$$

System (12), (13) is a simplified model problem derived from the BSSN system under the linearization assumption (with unit lapse and zero shift) by taking a second order reduction in time (see [2] for detail). Here, κ_{ij} plays the role of perturbation of the extrinsic curvature and M_i is the linearized momentum constraint.

One can quickly check that M_j obeys the subordinate wave equation,

$$\partial_t^2 M_i = \partial^l \partial_l M_i \tag{14}$$

Thus, constraint-compatible boundary conditions has to be specified for (12), (13).

Similarly to the previous section, we reduce (12), (13) to a constrained problem evolving (14) trivially. Namely, we introduce the equations

$$\partial_t^2 \kappa_{ij} = \partial^l \partial_l \kappa_{ij} - 2\partial_{(i} M_{j)} + \frac{1}{2}\delta_{ij}\partial^l M_l \tag{15}$$

subject to the constraint (13).

Substituting definition of M_j and using (14), one verifies that

$$\partial_t^2 M_i \equiv 0 \tag{16}$$

for any solution of (14).

Boundary conditions for (15) follow from the properties of the spatial differential operator. To letter can be studied by first, rewriting it in terms of κ_{ij} completely:

$$\partial_t^2 \kappa_{ij} = \partial^l \partial_l \kappa_{ij} - 2\partial_{(i}\partial^l \kappa_{|l|j)} + 2\partial_i \partial_j k + \frac{1}{2}\delta_{ij}\partial^l \partial^m \kappa_{lm} - \frac{1}{2}\delta_{ij}\partial^l \partial_l \kappa^m_{\ m}. \tag{17}$$

Let us denote by $\mathbb{T}$ the space of all triple-indexed arrays $\omega_{pqr} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ which are symmetric in the last two indices, $\omega_{pqr} = \omega_{prq}$. Notice that at each point, the gradient $\partial_p \kappa_{qr} \in \mathbb{T}$. We introduce the linear algebraic operator $L^{pqr}_{lij} : \mathbb{T} \to \mathbb{T}$ by the formula

$$L^{pqr}_{lij} = \delta^p_l \delta^q_{(i}\delta^r_{j)} - \delta^p_{(i}\delta^q_{|l|}\delta^r_{j)} - \delta_{l(i}\delta^r_{j)}\delta^{pq} + 2\delta_{l(i}\delta^p_{j)}\delta^{qr} + \frac{1}{2}\delta_{ij}\delta^{pq}\delta^r_l - \frac{1}{2}\delta_{ij}\delta^p_l\delta^{qr}.$$

In terms of the operator L^{pqr}_{lij} which we call *the algebraic operator of the second order equation*, (17) can be restated in the divergence form

$$\partial_t^2 \kappa_{ij} = \partial^l L^{pqr}_{lij} \partial_p \kappa_{qr}. \tag{18}$$

Theorem 4. The algebraic operator $L^{pqr}_{lij} : \mathbb{T} \to \mathbb{T}$ has eigenvalues 0, $3/2$, and $1/2$, with the multiplicities, correspondingly, 10, 5, 3, and full set of linearly independent eigenvectors, mutually orthogonal with the exception of three pairs (corresponding to eigenvalues 0 and $1/2$).

The fact that not all eigenvectors are mutually perpendicular implies that L^{pqr}_{lij} is not symmetric. Therefore, no straightforward first order symmetric hyperbolic reduction of (17) is possible. (It is also unlikely that a symmetrizer can be constructed for this problem). However, an energy estimate for (17) can be obtained using the following methodology.

Notice that the spectral structure of L^{pqr}_{lij} suggests to split gradient of κ_{ij} into the four components:

$$\partial_p \kappa_{qr} = \lambda_{pqr} + \mu_{pqr} + \eta_{pqr} + \rho_{pqr}, \tag{19}$$

where (denoting $\kappa = \kappa^i_i$)

$$\begin{aligned}
\lambda_{pqr} &= \frac{2}{3}[\partial_p \kappa_{qr} - \partial_{(q}\kappa_{r)p} - \frac{1}{2}\delta_{p(q}\partial^s \kappa_{|s|r)} \\
&\quad + \frac{1}{2}\delta_{p(q}\partial_{r)}\kappa + \frac{1}{2}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{2}\delta_{qr}\partial_p\kappa], \\
\mu_{pqr} &= \frac{1}{3}[\partial_p \kappa_{qr} + 2\partial_{(q}\kappa_{r)p} - \frac{4}{5}\delta_{p(q}\partial^s \kappa_{|s|r)} \\
&\quad - \frac{2}{5}\delta_{p(q}\partial_{r)}\kappa - \frac{2}{5}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{5}\delta_{qr}\partial_p\kappa], \\
\eta_{pqr} &= -[\delta_{p(q}\partial^s\kappa_{|s|r)}] + 3[\delta_{p(q}\partial_{r)}\kappa], \\
\rho_{pqr} &= \frac{1}{5}[8\delta_{p(q}\partial^s\kappa_{|s|r)} - \delta_{qr}\partial^s\kappa_{sp}] - \frac{2}{5}[8\delta_{p(q}\partial_{r)}\kappa - \delta_{qr}\partial_p\kappa].
\end{aligned}$$

The next theorem summarizes properties of decomposition (19).

Theorem 5. Let fields λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} be defined by (19), then

(a) At each point, fields λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} are eigenvectors of L^{pqr}_{lij} with eigenvalues, respectively, $3/2$, 0, $1/2$, and 0;

(b) fields λ_{pqr} satisfy the cyclic identity $\lambda_{pqr} + \lambda_{qrp} + \lambda_{rpq} = 0$, in addition, λ_{pqr} is trace free with respect to second and third indices, that is $\delta^{qr}\lambda_{pqr} = 0$. From these two properties one can deduce that λ_{pqr} is also trace free in first and third index, that is $\delta^{pr}\lambda_{pqr} = 0$. Fields μ_{pqr} are trace free in both pairs of indices pq and qr.

(c) the following table summarizes the mutual orthogonality of λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} with respect the to usual scalar product, $\langle \nu_{pqr}, \sigma_{pqr} \rangle = \nu_{pqr}\sigma^{pqr}$:

	λ	μ	η	ρ
λ	$\parallel$	$\perp$	$\perp$	$\perp$
μ	$\perp$	$\parallel$	$\perp$	$\perp$
η	$\perp$	$\perp$	$\parallel$	$\not\perp$
ρ	$\perp$	$\perp$	$\not\perp$	$\parallel$

Because of the trace free and cyclic properties, which are purely algebraic properties of the corresponding fields, the orthogonality between the fields remains even if one replaces κ_{ij} with different matrix fields in definitions of λ, μ, η, and ρ. Thus, decomposition (19) should be thought rather as a projection on different subspaces of the gradient. This important observation will simplify our calculations essentially when deriving energy estimates.

Substituting decomposition (19) and using Theorem 5 part (c), we rewrite (18) as

$$\partial_t^2 \kappa_{ij} = \partial^l (\frac{3}{2}\lambda_{lij} + \frac{1}{2}\eta_{lij}). \qquad (20)$$

To formulate the boundary conditions, we need to introduce some more notations: let vectors n_i, m_i, and l_i constitute an orthonormal triple in $\mathbb{R}^3$, the following vectors form the basis in the space of all symmetric matrices $\mathbb{S}$

$$e1_{ij} = \sqrt{2}\, n_{(i}m_{j)}, \quad e2_{ij} = \sqrt{2}\, n_{(i}l_{j)}, \quad e3_{ij} = \sqrt{2}\, m_{(i}l_{j)},$$

$$e4_{ij} = \frac{1}{\sqrt{2}}(m_i m_j - l_i l_j), \quad e5_{ij} = n_i n_j, \quad e6_{ij} = \frac{1}{\sqrt{2}}(m_i m_j + l_i l_j).$$

Let each of the constants $a1, \ldots, a5$ be either 0 or 1, we introduce the projection operator $(P_5)_{ij}^{pq} : \mathbb{S} \to \text{span}\{e1, e2, e3, e4, e5\} \subset \mathbb{S}$ by the formula

$$(P_5)_{ij}^{pq} = a1\, e1^{pq} e1_{ij} + \cdots + a5\, e5^{pq} e5_{ij}.$$

Let n_i be the outward unit normal to the boundary $\partial\Omega$, the following (homogeneous) condition will be imposed on fields κ_{ij} at the boundary,

$$(P_5)_{ij}^{pq} \partial_t \kappa_{pq} + 3n^p \lambda_{pij} + n^p \eta_{pij} = 0, \qquad \text{on } \partial\Omega. \tag{21}$$

Here we assume that the vector of coefficients $a = (a1, \ldots, a5)$ consists of 0s and 1s which may or may not change within the face. The value 0 corresponds to the Neumann type data on the corresponding components, while 1 gives radiative type data.

Theorem 6. Let κ_{ij} be a (sufficiently smooth) solution to (17) ((18), or (20)) satisfying boundary condition (21) (for either pair (a, b)). Then the solution κ_{ij} is unique and on the time interval $0 \le t \le T$ satisfies the energy estimate

$$\begin{aligned} &\|\partial_t \kappa_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s \kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2 \\ &\quad \le \quad \mathbf{e}^{3T/2}[\|\partial_t \kappa_{ij}(0)\|^2 + \frac{3}{2}\|\lambda_{lij}(0)\|^2 + \|\partial^s \kappa_{sj}(0) - 3M_j(0)\|^2 \\ &\qquad\qquad + 9\|M_j\|^2(0)] + \int_0^T F(t)\mathbf{e}^{3(T-t)/2} dt. \end{aligned} \tag{22}$$

where $F(t) = \frac{45}{4}\|M_j(0)\|^2 + (\frac{51}{4} + \frac{45}{2}t)\|\partial_t M_j(0)\|^2$.

Proof is available upon request.

Note that only the most directly related papers are cited below. The full list of appropriate citations is available from the author.

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