Modifying the Einstein Equations Off the Constraint Hypersurface

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JDB and Lisa L. Lowe
Phys Rev D74:104023, 2006

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The Problem: Most formulations of Einstein's equations suffer from constraint violating modes. These are solutions of the evolution equations that rapidly diverge from the constraint hypersurface. Constraint violating modes appear as nontrivial solutions of the *constraint evolution equations*.

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The Problem: Most formulations of Einstein's equations suffer from constraint violating modes. These are solutions of the evolution equations that rapidly diverge from the constraint hypersurface. Constraint violating modes appear as nontrivial solutions of the *constraint evolution equations*.

The Solution: Modify the *Einstein evolution equations* by adding terms proportional to the constraints. Do this in such a way that constraint violating modes no longer appear in the constraint evolution equations.

Einstein evolution equations (*gdot–Kdot* version):

$$\begin{aligned}
\partial_\perp g_{ab} &= -2\alpha K_{ab} \\
\partial_\perp K_{ab} &= \alpha(K K_{ab} - 2K_{ac}K^c_b + R_{ab}) - D_a D_b\,\alpha
\end{aligned}$$

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Constraints:

$$\begin{aligned} \mathcal{H} &\equiv K^2 - K_{ab}K^{ab} + R \\ \mathcal{M}_a &\equiv D_b K^b_a - D_a K \end{aligned}$$

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Constraint evolution equations:

$$\begin{aligned} \partial_\perp \mathcal{H} &= 2Kg^{ab}\partial_\perp K_{ab} + \cdots \\ &= 2Kg^{ab}\big[\alpha(KK_{ab} - 2K_{ac}K^c_b + R_{ab}) - D_a D_b \alpha\big] + \cdots \end{aligned}$$

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$$\begin{aligned} \partial_\perp \mathcal{H} &= 2\alpha K \mathcal{H} - 2\alpha D_a \mathcal{M}^a - 4\mathcal{M}^a D_a \alpha \\ \partial_\perp \mathcal{M}_c &= \alpha K \mathcal{M}_c - \mathcal{H} D_c \alpha - \alpha D_c \mathcal{H}/2 \end{aligned}$$

Modify the Einstein evolution equations off the constraint hypersurface by adding linear functions of the constraints and their spatial derivatives:

$$
\begin{aligned}
\partial_\perp g_{ab} &= -2\alpha K_{ab} + \Phi_{ab}(\mathcal{H}, \mathcal{M}) \\
\partial_\perp K_{ab} &= \alpha(K K_{ab} - 2K_{ac}K^c_b + R_{ab}) - D_a D_b\,\alpha + \Psi_{ab}(\mathcal{H}, \mathcal{M})
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This leads to new constraint evolution equations:

$$
\begin{aligned}
\partial_\perp \mathcal{H} &= 2\alpha K\mathcal{H} - 2\alpha D_a\mathcal{M}^a - 4\mathcal{M}^a D_a\alpha + \frac{\delta\mathcal{H}}{\delta g_{ab}}\Phi_{ab} + \frac{\delta\mathcal{H}}{\delta K_{ab}}\Psi_{ab} \\
\partial_\perp \mathcal{M}_c &= \alpha K\mathcal{M}_c - \mathcal{H}D_c\alpha - \alpha D_c\mathcal{H}/2 + \frac{\delta\mathcal{M}_c}{\delta g_{ab}}\Phi_{ab} + \frac{\delta\mathcal{M}_c}{\delta K_{ab}}\Psi_{ab}
\end{aligned}
$$

Key Question: How do we choose the functions $\Phi_{ab}(\mathcal{H}, \mathcal{M})$ and $\Psi_{ab}(\mathcal{H}, \mathcal{M})$ such that the constraint evolution equations are well behaved?

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Trial–and–error approach:

- L.E. Kidder, M.A. Scheel, and S.A. Teukolsky (Phys.Rev.D64:064017,2001)
- H. Shinkai and G. Yoneda (Class.Quant.Grav.19:1027-1050,2002)
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Our approach: Specify the desired right–hand sides for $\partial_\perp \mathcal{H}$ and $\partial_\perp \mathcal{M}_a$ then *solve* for Φ_{ab} and Ψ_{ab}.

For example: Choose the new constraint evolution equations to be

$$\begin{aligned}\partial_\perp \mathcal{H} &= -c_1 \mathcal{H} \\ \partial_\perp \mathcal{M}_a &= -c_2 \mathcal{M}_a\end{aligned}$$

Then we must find Φ_{ab} and Ψ_{ab} that satisfy

$$\begin{aligned}\frac{\delta \mathcal{H}}{\delta g_{ab}} \Phi_{ab} + \frac{\delta \mathcal{H}}{\delta K_{ab}} \Psi_{ab} &= -2\alpha K \mathcal{H} + 2\alpha D_a \mathcal{M}^a + 4\mathcal{M}^a D_a \alpha - c_1 \mathcal{H} \\ \frac{\delta \mathcal{M}_c}{\delta g_{ab}} \Phi_{ab} + \frac{\delta \mathcal{M}_c}{\delta K_{ab}} \Psi_{ab} &= -\alpha K \mathcal{M}_c + \mathcal{H} D_c \alpha + \alpha D_c \mathcal{H}/2 - c_2 \mathcal{M}_c\end{aligned}$$

The coefficients of Φ_{ab} and Ψ_{ab} are differential operators (Fréchet derivatives):

$$\begin{aligned}
\frac{\delta \mathcal{H}}{\delta g_{ab}} &= D^a D^b - g^{ab} D^c D_c + \cdots \\
\frac{\delta \mathcal{H}}{\delta K_{ab}} &= \cdots \\
\frac{\delta \mathcal{M}_c}{\delta g_{ab}} &= \frac{1}{2}(\delta_c^d K^{ab} + g^{ab} K_c^d) D_d - K_c^{(a} D^{b)} + \cdots \\
\frac{\delta \mathcal{M}_c}{\delta K_{ab}} &= \delta_c^{(a} D^{b)} - g^{ab} D_c
\end{aligned}$$

So we have 4 coupled differential equations

$$[D^a D^b - g^{ab} D^c D_c + \cdots]\Phi_{ab} + [\cdots]\Psi_{ab}$$
$$= \text{terms linear in } \mathcal{H} \text{ and } \mathcal{M}_c$$
$$\left[(\delta_c^d K^{ab} + g^{ab} K_c^d) D_d/2 - K_c^{(a} D^{b)} + \cdots\right] \Phi_{ab}$$
$$+ [\delta_c^{(a} D^{b)} - g^{ab} D_c]\Psi_{ab}$$
$$= \text{terms linear in } \mathcal{H} \text{ and } \mathcal{M}_c$$

for the 12 unknowns Φ_{ab} and Ψ_{ab}.

$\exists$ many solutions.

Choose the unknowns such that the differential equations form a nice elliptic system. Here's one way:

$$
\begin{aligned}
\Phi_{ab} &= 4g_{ab}\hat{\Phi} \\
\Psi_{ab} &= D_a\hat{\Psi}_b + D_b\hat{\Psi}_a - \frac{2}{3}g_{ab}D_c\hat{\Psi}^c - 2(K_{ab} - Kg_{ab})\hat{\Phi}
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The new unknowns are $\hat{\Phi}$ and $\hat{\Psi}_a$.

The differential equations become:

$$-8D^cD_c\hat{\Phi} + \text{non–principal terms}$$
$$= \text{terms linear in } \mathcal{H} \text{ and } \mathcal{M}_c$$
$$D^cD_c\hat{\Psi}_a + \frac{1}{3}D_aD^c\hat{\Psi}_c + \text{non–principal terms}$$
$$= \text{terms linear in } \mathcal{H} \text{ and } \mathcal{M}_c$$

Summary: Einstein evolution equations replaced by a mixed "hyperbolic–elliptic" system:

$$\partial_\perp g_{ab} = -2\alpha K_{ab} + 4 g_{ab}\hat{\Phi}$$

$$\partial_\perp K_{ab} = \alpha(K K_{ab} - 2K_{ac}K^c_b + R_{ab}) - D_a D_b\,\alpha + D_a\hat{\Psi}_b + D_b\hat{\Psi}_a - \frac{2}{3} g_{ab} D_c \hat{\Psi}^c - 2(K_{ab} - K g_{ab})\hat{\Phi}$$

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- This scheme is computationally similar to constrainted evolution. (The elliptic sector is equivalent to the linearized constraints.)
- This scheme is different from constrained evolution. We are not controlling the *value* of the constraints. Rather, we are controlling the *time evolution* of the constraints. This is done by adding linear, nonlocal functions of the constraints to the right–hand sides of the Einstein evolution equations.
- Is the full "hyperbolic–elliptic" system well–posed?

Numerical Tests:

- 3-D grid, periodic identification
  - Advantage: no boundary conditions to worry about.
  - Disadvantage: Can be harder to solve the elliptic problem. For example, $D^2\varphi = \rho$ has no solution unless $\int d^3x\, \rho = 0$ (Maximum principle).

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- Pseudospectral collocation
- Fourier basis functions
- Fourth order Runge–Kutta
- GMRES elliptic solver

TEST#1

Initial Data:

$$
\begin{aligned}
K_{ab} &= 0.2 + \text{random noise} \sim 10^{-10} \\
g_{ab} &= \delta_{ab} + \text{random noise} \sim 10^{-10}
\end{aligned}
$$

Gauge:

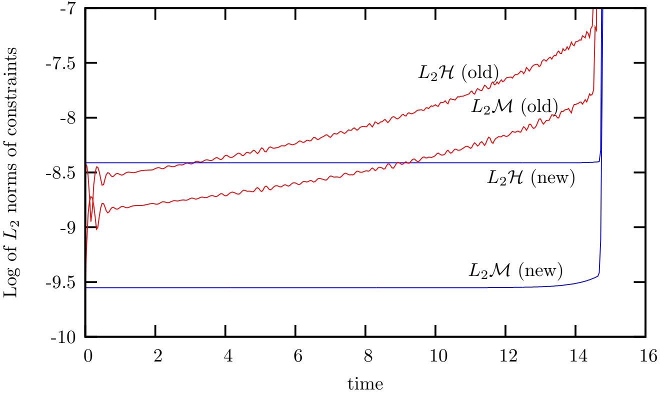
$$\alpha = 1 \ , \qquad \beta^a = 0$$

Red :

$$\begin{cases} \partial_\perp \mathcal{H} = 2\alpha K \mathcal{H} - 2\alpha D_a \mathcal{M}^a - 4\mathcal{M}^a D_a \alpha \\ \partial_\perp \mathcal{M}_c = \alpha K \mathcal{M}_c - \mathcal{H} D_c \alpha - \alpha D_c \mathcal{H}/2 \end{cases}$$

Blue :

$$\begin{cases} \partial_\perp \mathcal{H} = 0 \\ \partial_\perp \mathcal{M}_c = 0 \end{cases}$$



Evolution ends in a "big crunch".

TEST#2: Gowdy T^3, harmonic slicing (crunch at $t \sim 20$)

Red :
$$\begin{cases} \partial_\perp \mathcal{H} = 2\alpha K \mathcal{H} - 2\alpha D_a \mathcal{M}^a - 4\mathcal{M}^a D_a \alpha \\ \partial_\perp \mathcal{M}_c = \alpha K \mathcal{M}_c - \mathcal{H} D_c \alpha - \alpha D_c \mathcal{H}/2 \end{cases}$$

Blue :
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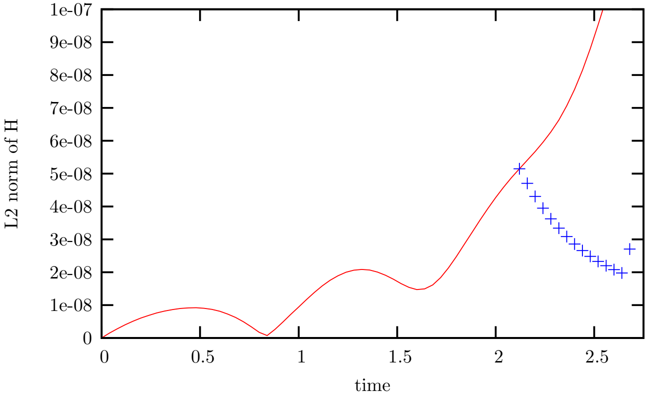

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- *etc* . . .