Towards Absorbing Outer Boundaries in General Relativity

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with

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Outline

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2. Perturbations on Minkowski
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   - Reflection coefficients
   - New boundary condition
3. Backscatter
   - Regge-Wheeler equation
   - Reflection coefficients
   - New boundary condition–improved(!?)
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1. Absorbing Outer Boundaries

Absorbing outer boundaries

Replace unbounded domain with a compact domain plus an artificial outer boundary.

Ideally, the artificial boundary is completely transparent to the physical problem on the unbounded domain.

Realistically, shoot for boundary conditions (b.c.'s) which:

1. Form a well-posed initial boundary value problem (IBVP).
2. Insure that very little spurious reflection of gravitational radiation occurs from the outer boundary.

Absorbing outer boundaries

General Relativity

A Challenging Problem!

- The future geometry of the outer boundary is not known *a priori*.
- Constraint modes propagate across the boundary.
- “Outgoing” and “ingoing” radiation is difficult to define because of nonlinearities and gauge freedom.

Absorbing outer boundaries

General Relativity

Constraint-preserving boundary conditions (CPBC) $\&$ b.c.'s on the gravitational radiation:

- Well-posed IBVP for Einstein's vacuum field equations.
  Friedrich and Nagy 1999
- CPBC $\& \; \partial_t \Psi_0 \hat{=} 0$ numerically implemented.
  Kidder et al. 2005, Sarbach and Tiglio 2005, Lindblom et al. 2005, Scheel et al. 2006, Rinne 2006
- Improved hierarchy of local b.c.'s on Ψ_0 introduced.
  LTB and O. Sarbach, CQG, **23**, 6709–6744 (2006) (this talk)

2. Perturbations on Minkowski

Minkowski: Bianchi equations

Assume:

Weak field gravity (outer boundary is in the wave zone).
Computational domain is a ball B_R of radius R.

- Metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$

  where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small ($|h_{\mu\nu}| \ll 1$) perturbation. Neglect quadratic and higher order terms in $h_{\mu\nu}$.

- Described by the vacuum Bianchi equations:

$$\nabla^a C_{abcd} = 0,$$

  where C_{abcd} is the linearized Weyl tensor.

Minkowski: Bianchi equations

- C_{abcd} vanishes on the background => it's invariant w.r.t. infinitesimal coordinate transformations => no gauge modes.
- Expand C_{abcd} in spherical tensor harmonics.
- Group the 10 components of C_{abcd} into 5 complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4, defined w.r.t. the null tetrad: $l = (\partial_t + \partial_r)/\sqrt{2}$, $k = (\partial_t - \partial_r)/\sqrt{2}$, m, $\bar{m}$.

Minkowski: Bianchi equations

Assume constraints are satisfied.

- For $\ell \geq 2$, C_{abcd} is determined entirely by the master equation

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_2(t,r) = 0.$$

- Exact analytic solutions are

$$\begin{aligned}\psi_{2\,\nwarrow,\ell}(t,r) &= \frac{1}{r^2}a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger V_\ell(r+t),\\ \psi_{2\,\nearrow,\ell}(t,r) &= \frac{1}{r^2}a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger U_\ell(r-t),\end{aligned}$$

  where $a_\ell \equiv \partial_r + \frac{\ell}{r}$ and $a_\ell^\dagger \equiv -\partial_r + \frac{\ell}{r}$.

- Relation between Ψ_0 and ψ_2:
  $r^5\Psi_0 \sim (b_-)^2\psi_2, \qquad b_- = r^2(\partial_t + \partial_r).$

Minkowski: Reflection coefficients

- In- and outgoing solutions are simply related by $t \mapsto -t$.
- Clear how to quantify amount of spurious reflection and define a reflection coefficient.
- Can show that along outgoing null geodesics ($t - r = const.$), our exact outgoing solutions satisfy

$$\Psi_j = O(r^{j-5}), \quad j = 0, 1, 2, 3, 4.$$

peeling theorem, Penrose, 1965.

Minkowski: Reflection coefficients

For b.c. $\partial_t \Psi_0 \hat{=} 0$:

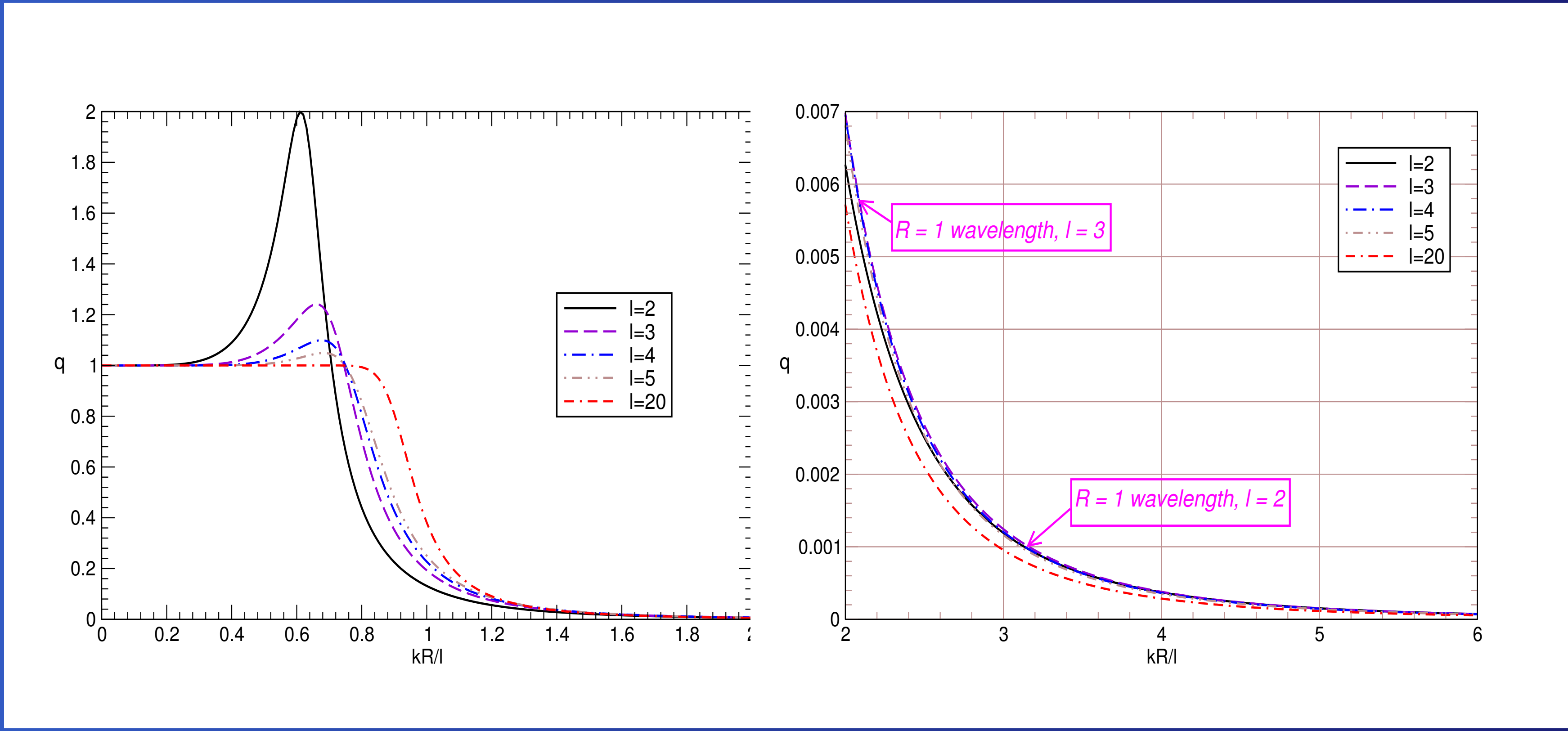
- Ψ_0 falls off as $1/r^5$ along the outgoing null radial geodesics.
- A solution to the IBVP corresponding to the b.c. $\partial_t \Psi_0 \hat{=} 0$ consists of a superposition of an out- and an ingoing wave.
- To quantify the amount of reflection, make the monochromatic ansatz

$$\psi_2(t,r) = \frac{1}{r^2}\, a_\ell^\dagger a_{\ell-1}^\dagger \ldots a_1^\dagger \left(e^{ik(r-t)} + \gamma e^{-ik(r+t)} \right),$$

where γ is an *amplitude reflection coefficient*.

Minkowski: Reflection coefficients

$q_{\text{ freeze}} \equiv |\gamma|$ is of order unity if $kR < \ell$, and decays as $(kR)^{-4}$ for large kR/ℓ.



Minkowski: New boundary condition

Hierarchy $\mathcal{B}_L$ of improved b.c.'s

- For $L \geq 2$: $\mathcal{B}_L$ is *perfectly absorbing* for linearized gravitational radiation in flat space with $\ell \leq L$.

$$\mathcal{B}_L : \quad (b_-)^{L-1}(r^5\Psi_0) \hat{=} 0\big|_{r=R} .$$

- Expect few lower multipoles to dominate in numerical simulations, so an implementation of this b.c. for $L = 2, 3$ or 4 should suppress much of the spurious reflection.
- For $L = 2$:

$$\mathcal{B}_2 : \quad (\partial_t + \partial_r)\partial_t(r^5\Psi_0) \hat{=} 0.$$

3. Backscatter

Backscatter: Regge-Wheeler equation

Recall assumptions:

Weak field gravity (outer boundary is in the wave zone).
Computational domain is a ball B_R of radius R.

- Small perturbations about the Schwarzschild metric with mass M, where M and outer boundary radius R.
- For odd-parity sector, the master equation for $Im(\Psi_2)$ is given by the Regge-Wheeler equation:

$$\left[\partial_t^2 - \partial_{r_*}^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)\right]\psi_2(t,r) = 0,$$

where r_* is the tortoise coordinate.

Backscatter: Regge-Wheeler equation

- Define new coordinates $\tau = t + r - r_*, \quad \rho = r,$.
- Compute first order corrections in $2M/R$ to the exact flat space outgoing solution:

$$\psi_2(\tau,\rho) = \frac{1}{\rho^2}\left[a_\ell(\rho)^\dagger a_{\ell-1}(\rho)^\dagger ... a_1(\rho)^\dagger U(\rho-\tau) + \sum_{k=1}^{\infty}\left(\frac{2M}{R}\right)^k g_k(\tau,\rho)\right]$$

=> $\left[\partial_\tau^2 - \partial_\rho^2 + \frac{\ell(\ell+1)}{\rho^2}\right] g_1(\tau,\rho) = S[U]$.

Backscatter: Regge-Wheeler equation

- First order correction:

$$g_1(\tau,\rho) = \frac{3R}{4\rho^4}U'(\rho-\tau) + \frac{R}{4\rho^2}\int\limits_{\rho-\tau}^{\infty} K_2(\tau,\rho,x)U(x)dx,$$

where the integral kernel K_2 is given by

$$K_2(\tau,\rho,x) \equiv \frac{3}{2\rho^4}\left[w^{-4} + 2w^{-3} + 2w^{-2}\right]_{w=\frac{\tau+\rho+x}{2\rho}}, \qquad x > \rho - \tau.$$

Backscatter: Regge-Wheeler equation

- Outgoing solution ($\ell = 2$) to first order in $\frac{2M}{R}$:

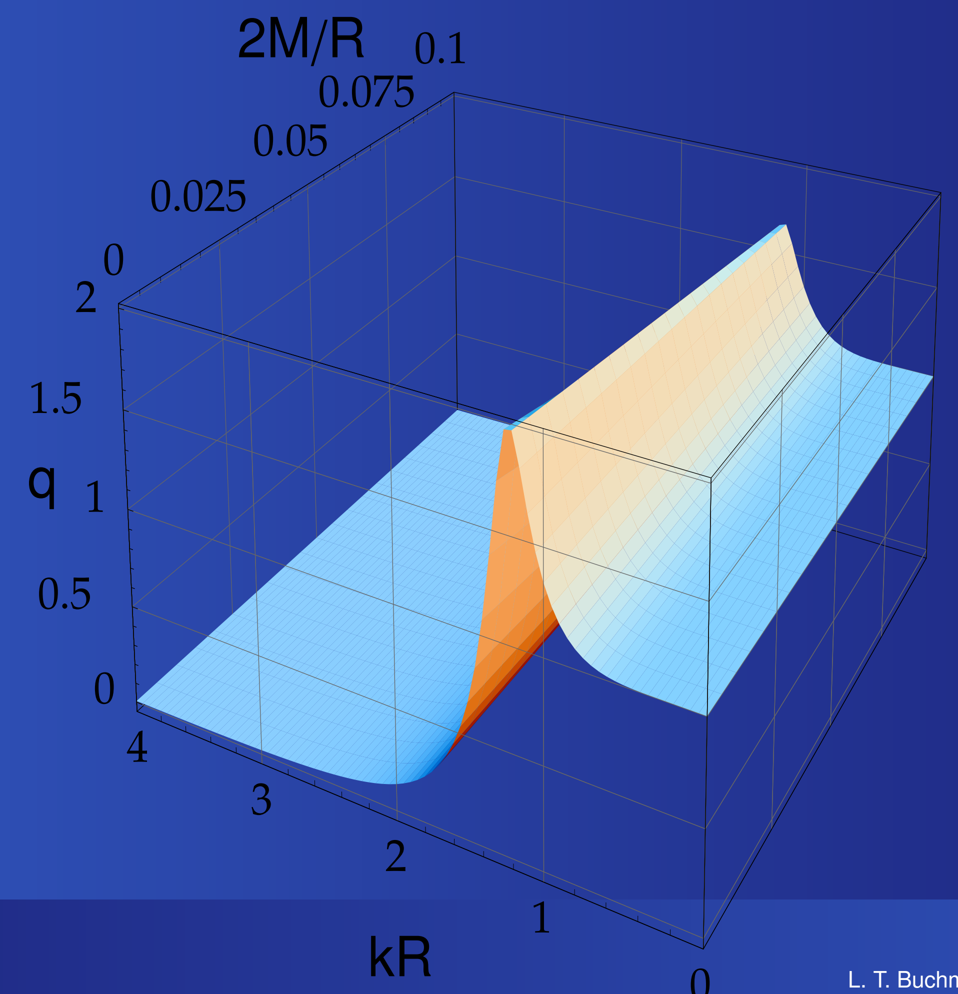
$$\psi_{2\nearrow}(t,r) = \frac{1}{r^2}U''(r_* - t) - \frac{3}{r^3}U'(r_* - t) + \frac{3}{r^4}U(r_* - t)$$
$$+\frac{2M}{R}\left[\frac{3R}{4r^4}U'(r_* - t) + \frac{R}{4r^2}\int\limits_{r_*-t}^{\infty} K_2(t + r - r_*, r, x)U(x)dx\right]$$

- Corresponding ingoing solution by reversing the sign of t.

Backscatter: Reflection coefficients

Result (quadrupolar radiation):

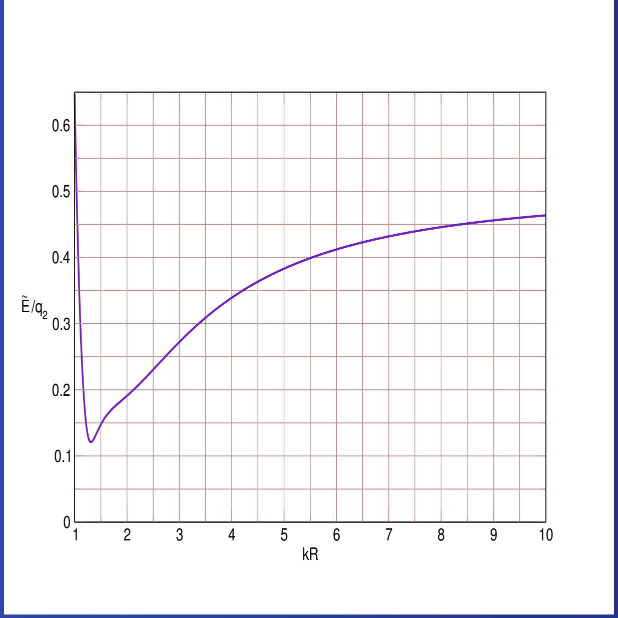
q_{freeze} depends only weakly on $2M/R$ (for $2M/R \ll 1$).



Backscatter: Reflection coefficients

For $\mathcal{B}_2$:

- $q_{\text{ new}} = \frac{2M}{R} E(kR) + \ldots$
- For large kR, $E(kR)$ decays as kR^{-4}.
- Plot the ratio $\frac{E(kR)}{q_{\text{ freeze}}}$:



- $q_{\text{ new}} < q_{\text{ freeze}}$ by a factor of M/R for $kR > 1.05$.

Backscatter: New b.c.–improved(!?)

- Recall: $\mathcal{B}_2 : \quad (\partial_t + \partial_r)\partial_t(r^5\Psi_0) \hat{=} 0$ is perfectly absorbing only if $M = 0$. We have an expression for the (small) amount of reflection when $M \neq 0$, to first order in $\frac{2M}{R}$.
- Ansatz: $\tilde{\mathcal{B}}_2 : \quad \left[(\partial_t + \partial_r)\partial_t + \frac{2M}{r}\hat{\mathbf{A}}\right](r^5\Psi_0) \hat{=} 0$.
- Task: Find $\hat{\mathbf{A}}$ so that $\tilde{\mathcal{B}}_2$ is perfectly absorbing.
- $\Psi_0 \approx U(r_* - t) + \frac{2M}{r}\left[\text{stuff} + \int\limits_0^\infty k(1+y)U(r_* - t + 2ry)dy\right]$.
- Plug Ψ_0 into $\tilde{\mathcal{B}}_2$ => $\hat{\mathbf{A}}$ is an integral operator with an infinite limit–no good for numerical relativity.
- Assume that U vanishes for $r > R$, and change upper limit on integral to $t/2r$.

Backscatter: New b.c.–improved(!?)

- Result: Non-local boundary condition (involves an integral operator).
- Errors due to an inadequate gauge may outweigh the improvement gained when numerically implementing $\tilde{\mathcal{B}}_2$. Therefore, at this point, we cannot guarantee that $\tilde{\mathcal{B}}_2$ gives an improvement over $\mathcal{B}_2$ in numerical simulations.

4. Conclusions

Conclusions

- Estimate amount of spurious reflection off an artificial outer boundary with the b.c. $\partial_t \Psi_0 \hat{=} 0$.
- Propose a hierarchy $\mathcal{B}_L$ $(L = 2, 3, 4, ...)$ of new local b.c.'s which are perfectly absorbing for linearized waves with $\ell \leq L$ on a flat background.
- Including backscatter (to 1st order), these new b.c.'s give a reflection coefficient which is smaller than the one for $\partial_t \Psi_0 \hat{=} 0$ by a factor of M/R for $kR > 1.05$.

Conclusions

Work in progress:

- Understand the practicality of $\tilde{\mathcal{B}}_2$, intended to be perfectly absorbing when 1st order corrections for backscatter are included.
- Numerical implementation and tests.
- Even parity sector (for backscatter calculations).
- More general outer boundary shapes.

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