On relativistic Euler equations

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OUTLINES

1. Background

2. Local Existence

3. Blowup: Finite energy case

4. Blowup: Infinite energy case

5. Remarks

1. Relativistic perfect Flows

Einstein equation of General Relativity:

$G^{ij} = kT^{ij}$

$G^{ij} =$ Einstein curvature tensor,

g_{ij} =metric,

$T^{ij} =$ stress-energy tensor.

Bianchi identities ($DivG^{ij} = 0$) implies the conservation of energy and momentum:

$$DivT^{ij} = 0. \qquad (0)$$

Consider a perfect fluid in 4-D Minkowski spacetime:

$T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij}.$

p:pressure, ρ: mass-energy density,

u^i: velocity, c: speed of light

$\eta_{ij} = diag(-1, 1, 1, 1)$: Minkowski metric.

1.1. Relativistic Euler equations

$$\begin{cases} \partial_t\left(\frac{\rho c^2+p}{c^2-v^2} - \frac{p}{c^2}\right) + \nabla_x \bullet \left(\frac{\rho c^2+p}{c^2-v^2} v\right) = 0 \\ \partial_t\left(\frac{\rho c^2+p}{c^2-v^2} v\right) + \nabla_x \bullet \left(\frac{\rho c^2+p}{c^2-v^2} v \otimes v\right) + \nabla_x \, p = 0, \\ \rho(x,0) = \rho_0(x), v(x,0) = v_0(x). \end{cases} \quad (1)$$

- $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $v = (v_1, v_2, v_3)$
- Equation of state: $p = p(\rho)$, $p(0) = 0$ with

$$p(\rho) \geq 0, \;\; 0 < p'(\rho) < c^2, \;\; p''(\rho) \geq 0, \;\; for \;\; \rho \in (\rho_*, \rho^*), \qquad (2)$$

where $0 \leq \rho_* < \rho^* \leq \infty$

1.2. Examples

• γ-law: $p(\rho) = \sigma^2 \rho^\gamma$ with $\gamma \geq 1$:

if $\gamma = 1$, then $\rho^* = \infty$;

if $\gamma > 1$, then $p'(\rho^*) = c^2$.

•neutron stars:

$$p = Ac^5 a(y), \ \rho = Ac^3 b(y),$$
$$a(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}}\, dq, \quad b(y) = 3\int_0^y q^2\sqrt{1+q^2}\, dq. \tag{3}$$

Here A is a positive constant.

1.3. Convenient Notations

Define

$$\tilde{\rho} = \frac{\rho c^2 + p}{c^2 - v^2}, \text{ and } \hat{\rho} = (\frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}).$$

Then

$$\hat{\rho} = \frac{1}{c^2}\tilde{\rho} v^2 + \rho, \text{ and } \tilde{\rho} = \frac{1}{c^2}\tilde{\rho} v^2 + \frac{p}{c^2} + \rho.$$

and the relativistic Euler equations become

$$\begin{cases} \hat{\rho}_t + \nabla_x \bullet (\tilde{\rho} v) = 0 \\ (\tilde{\rho} v)_t + \nabla_x \bullet (\tilde{\rho} v \otimes v) + \nabla_x p(\rho) = 0, \\ \rho(x,0) = \rho_0(x), \ v(x,0) = v_0(x). \end{cases} \tag{4}$$

1.3. Main Results

We shall discuss two different cases:

Finite initial energy: If the initial data has compact support, the life-span of any non-trivial smooth solution is finite.

Infinite initial energy: If the initial data is a compactly supported perturbation around a non-vacuum background, then the life-span of smooth solutions is finite when the radial component of the initial momentum is large.

—A. Rendall (1992), Guo and Tahvilda-Zadeh (1999).

1.4. Approach via averaged quantities

For non-relativistic compressible flows:

–Sideris, 1985.

–Makino, Ukai, Kawashima, 1986,

–Markino, Perthame, 1990,

–Chemin, 1990,

–Xin, 1998.

...

2. Local Existence of Solutions

⋆ Theory of Symmetric hyperbolic systems:

Friedrich-Kato-Majda

⋆ Convex entropy method

Godunov, Dafermos.

2.1. Domain

Let $\rho_* < \rho^*$ be non-negative constants in (2) subject to subluminal condition $p'(\rho^*) \leq c^2$. We set

$$z = (\rho, v_1, v_2, v_3)^T$$

and define the region Ω_z by

$$\Omega_z = \{z : \rho_* < \rho < \rho^*, v^2 < c^2\}.$$

2.2. Theorem 1 (Makino-Ukai, Pan-Smoller)

Suppose $z_0(x) = (\rho_0(x), v_0(x))^T$ is smooth, taking values in any compact subset $\mathbf{D}$ of Ω_z and that $\nabla_x z_0(x) \in H^l(\mathbf{R}^3)$ for some $l > 3/2$. Then there exists T_∞, $0 < T_\infty \leq \infty$, and a unique $z(x,t) = (\rho(x,t), v(x,t))^T$ on $\mathbf{R}^3 \times [0, T_\infty)$, taking values in Ω_z, which is a classical solution of the Cauchy problem (4) on $\mathbf{R}^3 \times [0, T_\infty)$. Furthermore,

$$\nabla_x z(\cdot, t) \in C^0([0, T_\infty); H^l). \tag{5}$$

The interval $[0, T_\infty)$ is maximal: whenever $T_\infty < \infty$,

$$\lim_{t \to T_\infty} \sup \|\nabla_x z(\cdot, t)\|_{L^\infty} = \infty \tag{6}$$

and/or the range of $z(\cdot, t)$ escapes from every compact subset of Ω_z as $t \to T_\infty$.

2.3 Construction of entropy

Rewrite (4) in the form of conservation laws,

$$\theta_t + \sum_{k=1}^{3} (f^k(\theta))_{x_k} = 0, \tag{7}$$

where $\theta = (\theta_0, \theta_1, \theta_2, \theta_3)^T$ and $f^k(\theta) = (\theta_k, f_1^k, f_2^k, f_3^k)^T$:

$$\begin{aligned} &\theta_0 = \widehat{\rho},\ \ \theta_j = \tilde{\rho} v_j, \\ &f_j^k = \tilde{\rho} v_j v_k + p\delta_{jk},\ \ j = 1, 2, 3. \end{aligned} \tag{8}$$

Entropy and Flux

The scalar function $\eta = \eta(\theta)$ is called an entropy function and scalar functions $q^k(\theta)$, $k = 1, 2, 3$ are called entropy flux functions, if they satisfy:

$$\nabla_\theta \eta(\theta) \nabla_\theta f^k(\theta) = \nabla_\theta q^k(\theta). \tag{9}$$

We will solve (9) keeping the mechanical energy of classical Euler equations in mind. Thus, instead of θ, we will use $z = (\rho, v_1, v_2, v_3)^T$ as independent variables.

$$\nabla_z \eta C^k = D_z q^k, \ k = 1, 2, 3, \tag{10}$$

Reduction

Formally, (10) is an over-determined system, consisting of 12 equations for 4 unknowns. We seek solutions with the special form:

$$\eta = \eta(\rho, y),\;\; q^k = Q(\rho, y) v_k,\;\; y = v^2 = v_1^2 + v_2^2 + v_3^2. \tag{11}$$

to reduce the number of equations in (10):

$$\begin{cases} \eta_y = Q_y, \\ c^2 C_1 \eta_\rho + 2C_2(1 - C_1 y)\eta_y = Q_\rho \\ C_3 \eta_\rho - 2C_4 y \eta_y = Q. \end{cases} \tag{12}$$

Entropy-flux pair

Define

$$\phi(\rho) = \int_{\rho_m}^{\rho} \frac{c^2}{rc^2 + p(r)}\, dr, \quad K = \rho_m c^2 + p(\rho_m),$$

ρ_m being any fixed number in (ρ_*, ρ^*).

Solve (12):

$$\eta = c^2 \hat{\rho} - \frac{cKe^{\phi(\rho)}}{\sqrt{c^2 - v^2}}. \tag{13}$$

The associated entropy-flux is $(q^1, q^2, q^3)^T$ defined by

$$q^k = \frac{c^2(\rho c^2 + p)}{c^2 - v^2} v_k - \frac{cKe^{\phi(\rho)}}{\sqrt{c^2 - v^2}} v_k. \tag{14}$$

3. Blowup in finite energy case

Assume that data has compact support

$$\rho_0(x) = 0, \quad v_0(x) = 0, \ for \ |x| \geq R.$$

Thm 2 (Pan&Smoller, 2006) The life-span of any nontrivial smooth solution of (4) with compactly supported initial data, is finite.

3.1. Invariance of support

Lemma 3.1 The support of the smooth solution of (4) with compactly supported initial data is invariant in time

Both the wave speed and the particle velocity are zero at the boundary of the support.

Proof of Theorem 2 :

$$H(t) = \frac{1}{2}\int \widehat{\rho}|x|^2 \, dx, \;\; F(t) = \int \tilde{\rho} v \cdot x \, dx, \;\; E(t) = \int \widehat{\rho} \, dx.$$

⋆ Conservation of energy: $E(t) = E(0) > 0$.

⋆ Upper bound of $H(t)$:

$$H(t) \leq \frac{1}{2} R^2 E(0). \tag{15}$$

⋆ Lower bound of $H''(t)$:

$$H''(t) \geq B > 0. \tag{16}$$

4. Blowup in infinite energy case

Assume that

$$\rho_0(x) = \bar{\rho} > 0, \quad v_0(x) = 0, \; for \; |x| \geq R.$$

Define

$$M(t) = \int \widehat{\rho}(\rho, v) - \widehat{\rho}(\bar{\rho}, 0) \; dx,$$

$$F(t) = \int \tilde{\rho} v \bullet x \; dx.$$

Thm 3 (Pan & Smoller, 2006) Let $M(0) > 0$, $F(0) > 0$, and $p'(\bar{\rho}) < \frac{c^2}{3}$. If $F(0)$ is large enough, the life-span of the smooth solution of (4) is finite.

4.1 Finite propagation speed

Lemma 4.1 The support of $(\rho - \bar{\rho}, v)(x,t)$ is contained in the ball $B(t) = \{x : |x| \leq R + st\}$ where

$$s = \sqrt{p'(\bar{\rho})}$$

is the sound speed in the far field.

- Consequence of Local energy estimates.

4.2. Proof of Theorem 3

- $M(t) = M(0) > 0$,
- $F'(t) \geq (1 - \frac{3s^2}{c^2})A^{-1}(R + st)^{-5}F^2(t)$.

Hence, $F(t)$ tends to infinity in finite time if

$$F(0) > (1 - \frac{3s^2}{c^2})^{-1}(4sAR^4). \tag{17}$$

5. Remarks

• In Theorems 2 and 3, the velocity in the far field is assumed to be zero initially. For more general case, say $v_0(x) = \bar{v}$ out of a bounded set, the change of variables (T. Sideris)

$$v \to v - \bar{v},\ x \to x + t\bar{v}$$

will reduce the problems into the case as in our Theorems.

• The condition

$$p'(\bar{\rho}) < \frac{c^2}{3}$$

in Theorem 3 rises naturally in the proof. Here, 3 is the space dimension. 3 becomes d for d space dimensions. In particular, for $d = 1$, this condition is that the sound speed is slower than the light speed. For $\gamma = 1$, $n = 1$, this condition guarantees the genuinely nonlinearity of the relativistic Euler equation; which allows the existence of global solutions in BV; see

– Smoller, Temple, 1993.

• The blowup results in Theorems 2 and 3 crucially depend on the compact support of the perturbations. The singularity formation for more general initial data remains open.

Nature of the singularities

• Type of singularities are not clear in Theorems 2 and 3.

—Shock formation, (Yes, for 1-d)

—v tends to c, or $p'(\rho) \to c^2$.

—concentration of the mass (Black Holes?)

• The singularity in Theorem 3 looks more like a shock formation.

Thank you!