On harmonic measures and group actions

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Definition 1. *A measure μ on M is a harmonic measure for the foliated space $(M, \mathcal{F})$ if*

$$\int \Delta f \cdot \mu = 0,$$

for every function f which is C^2 along the leaves and such that $\Delta f = 0$.

Unlike invariant measures for foliated spaces, harmonic measures are good because of the following fundamental theorem:

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There is a variety of characterization of harmonic measures. One of them is the following local structure theorem:

Theorem. [Garnett] *Locally in a foliation chart $D \times T$ for the foliated space $(M, \mathcal{F})$, a harmonic measure μ decomposes as*

$$\mu = h(x,t)\,\mathrm{vol}(x) \otimes \nu(t)$$

where "vol" is the Riemannian volume form on the leaves, ν is a measure on the *transversal T, and h is a measurable function such that $x \mapsto h(x,t)$ is a harmonic function for ν-almost all t.*

Group actions

Many examples of foliated spaces are given by group actions or homogeneous spaces of such actions. Let G be a (connected) Lie group and K a compact subgroup. If G acts continuously on a space M and the action is locally free, then M is foliated with leaves the orbits of G, which are homogeneous spaces of G of the form $\mathrm{Stab}(x)\backslash G$, and the quotient space M/K is foliated by double coset spaces of the form $\mathrm{Stab}(x)\backslash G/K$.

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But in general the orbits of a Lie group action have different dimension.

Lie Groups and Harmonic Functions

Let G be a (connected) Lie group, K is a compact subgroup, and X is the homogeneous space G/K. A function f on X (or a K-invariant function on G) is **harmonic** if $Dh = 0$ for all differential operators D on X which vanish on the constant functions and which are G-invariant in the sense that it commutes with the action of G.

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Example. *If $G = SL(2, \mathbf{R})$, $K = SO(2)$ then $X = G/K$ is the hyperbolic plane. The hyperbolic Laplacian Δ on X is an example of an invariant differential operator and any other invariant differential operator is a polynomial on Δ. Harmonic functions on G are the lifts of the usual harmonic functions of complex analysis.*

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An important result on harmonic functions is due to Godement and it generalizes the well-known mean value property for harmonic functions on the complex plane.

Theorem. [Godement] *A function h on X (or K-invariant on G) is harmonic if it is locally integrable and*

$$\int_K h(gk \cdot x) \cdot dk = h(g \cdot o)$$

where dk is Haar measure on X and o is the canonical base point of X.

Suppose that G acts on a space M and that μ is a measure on M which is quasi-invariant under this action. That is, μ and $g_*\mu$ have the same sets of measure zero. Then there is the Jacobian cocycle $j : M \times G \to \mathbf{R}^*$ which is given by the Radon-Nykodym derivative

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Question. *If G acts continuously on a compact space M, does M admit a harmonic probability measure?*

If so, what can be done with such measures? what dynamic, ergodic, geometric properties do they have?

Semisimple Lie groups

If G is a semisimple Lie group with finite center then G admits a product decomposition $G = KAN$ (Iwasawa decomposition): K is a maximal compact subgroup, the subgroup A is commutative, the subgroup N is nilpotent, and the product AN is solvable, connected and simply connected. This decomposition means that each element g of G can be written in a unique way as the product $g = kan$, where $k \in K$, $a \in A$ and $n \in N$.

If M is the centralizer of the Lie algebra of A in K, then the group $P = MAN$ is called the minimal parabolic subgroup of G. It is a solvable group (hence amenable) and $B = G/P = K/M$ is called the **maximal boundary** of the symmetric space $X = G/K$. It is one of the Satake-Furstenberg compactifications of the symmetric space X.

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The example to keep in mind is $G = \mathrm{SL}(n, \mathbf{R})$, $K = \mathrm{SO}(n)$, A the diagonal subgroup, N the subgroup of upper-triangular matrices with 1 on the diagonal. In this case P is the group of upper triangular matrices in G and G/K can be identified with the space of positive definite matrices of determinant 1.

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Invariant measures for a G-space M are harmonic. Another important example is the following:

Example. *Haar measure on K induces a measure β on $B = G/P = K/M$. It is only quasi-invariant under the canonical action of G, and its Radon-Nykodym derivative*

$$P(b, g) = \frac{dg_*^{-1}m}{dm}(b)$$

is the so called Poisson Kernel of B. The function $g \mapsto P(b, g)$ is harmonic on G for all $b \in B$ and the measure β is harmonic.

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Let β be the canonical measure on the boundary $B = G/P$ and pick a probability measure β' on G projecting to β. The mapping

$$\pi \mapsto \pi * \beta'$$

establishes that bijection, where $\pi * \beta'$ is the convolution

$$\int_M f(x) \cdot (\pi * \beta')(x) = \int_G \left(\int_M f(xg) \cdot \pi(x) \right) \cdot \beta'(g).$$

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If $(N, \mathcal{F})$ is a compact Riemann surface lamination whose leaves are hyperbolic, then the natural 3-dimensional foliation of the unit tangent bundle (for the hyperbolic metric) of N is given by a locally free action of $G = \mathrm{PSL}(2, \mathbf{R})$. A recent preprint of Bakhtin and Martinez establishes a bijective correspondence between harmonic measures μ on N and P-invariant measures π on $T_1 N$.

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Furstenberg, and others, have studied the concept of stationary measures for group actions which turns out to be related to the concept described here, and essentially equivalent for actions of semisimple Lie groups.

Suppose that G acts on M and that ν is an admissible probability measure on G. Given a measure μ on M we can define the convolution $\nu * \mu$ which is the

measure on M acting on functions by

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Theorem. [Furstenberg] *Suppose that ν is an admissible measure on G. If G acts continuously on a compact space M, then M admits a ν-stationary measure.*

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Furstenberg proved an structure theorem for stationary measures. This theorem was based on his study of the boundary of symmetric spaces and the integral representation of bounded ν-harmonic functions. (Note that the Radon-Nykodym cocycle is harmonic, but only bounded in trivial cases.)

If G is semisimple with finite center and ν is an admissible measure on G, then there is a compact homogeneous space $B_\nu = G/P_\nu$ where a Poisson formula for bounded ν-harmonic functions holds with respect to a K-invariant ν-harmonic measure β_ν. This B_ν is a covering space of the minimal boundary B.

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Theorem. [Furstenberg] *Let G be semisimple with finite center, maximal compact K, and admissible measure ν. Let B_ν be the boundary associated to ν as above and β'_μ a probability measure on G projecting to β_μ. If M a continuous G-space, then the mapping $\pi \mapsto \pi * \beta'_\nu$ establishes an affine bijection between P_ν-invariant probability measures π on M and ν-stationary measures on M.*

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Mixing is an ergodic property which invariant measures may or may not have and is shown to be equivalent to the vanishing of coefficients of the Koopman representation of the group by the action on $L^2(X, \mu)$ given by

$$\Pi(g)(f)(x) = f(xg).$$

To say that the action of P is mixing means that $p \mapsto \langle f_1, \Pi(p) f_2 \rangle$ converges to 0 as $p \longrightarrow \infty$ in P.

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In general, if X is a G-space with quasi-invariant measure μ and Radon-Nykodym cocycle $j(x, g)$, the Koopman representation is given by

$$\Pi(g)(f)(x) = f(xg)j(x, g)^{1/2}$$

In general, if X is a G-space with quasi-invariant measure μ and Radon-Nykodym cocycle $j(x,g)$, the Koopman representation is given by

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There is a concept in ergodic theory called *weakly mixing*, which is intermediate between ergodic and mixing for invariant measures, but in general not equivalent to any of them.

Definition 3. *Suppose that G acts on M with quasi-invariant measure μ. The action is weakly mixing if the Koopman representation of G on $L^2(M,\mu)$ has no finite dimensional invariant subspaces.*

(Note that ergodicity of an invariant measure means that the action has no invariant one-dimensional subspaces other than the constants.)

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(Note that ergodicity of an invariant measure means that the action has no invariant one-dimensional subspaces other than the constants.)

Theorem 3. *Let G be semisimple with finite center and no compact factors. Let M be a G-space with probability harmonic measure μ. If μ is ergodic and not invariant, then the action of G on M is weakly mixing.*

Structure of the Jacobian

The positive harmonic function $j(x, \cdot)$ on G has an integral representation

$$j(x, g) = \int_{B(G)} P(b, g) \cdot \mu_x(b)$$

where μ_x is a probability measure on the boundary $B(G)$ of G and P is the Poisson kernel.

Theorem 4. *Let G be semisimple with finite center and no compact factors. Let B be its maximal boundary and β the canonical harmonic measure on B. Suppose that μ is an ergodic harmonic measure for the G-space M. Then in the integral representation $\mu \equiv \{\mu_x\}$ above*

- *If μ is invariant, then $\mu_x = \beta$ for all x*

- *If μ is not invariant, then μ_x and β are mutually singularfor almost all x.*

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The assignment $J : M \to \mathcal{M}_1(B)$ given by $J(x) = \mu_x$ defines an equivariant map of M into the space of probability measures on B (but the action of G on B is not the standard one but a twisted one). The image JM is a G-space for this action, and the push forward measure $J_*\mu$ is a harmonic measure on it.

The G-space $\mathcal{M}_1(B)$ can be equivariantly identified with $\mathcal{H}_1(G)$ (the space of positive harmonic functions h on G such that $h(e) = 1$) where $g \in G$ acts on $h \in \mathcal{H}_1(G)$ via

$$(h \cdot g)(g') \mapsto \frac{h(gg')}{h(g)}.$$

The evaluation map ℓ given by

$$\ell : (h, g) \in \mathcal{H}_1(G) \times G \mapsto \ell(h, g) = h(g) \in \mathbf{R}_+^*$$

is a cocycle for this action. It is in fact a universal cocycle for Jacobians of harmonic measures on G-spaces.

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Theorem 5. *For any harmonic measure μ on any G-space M, the map $M \to JM$ is G-equivariant and the push forward measure $J_*\mu$ on $JM \subset B(G)$ has cocycle ℓ.*

Corollaries of these results on the structure of the Jacobian cocycle and the weakly mixing property are:

Corollary 1. *Let G be semisimple with finite center and no compact factors. Suppose that M is a G-space with a harmonic probability measure μ which is ergodic but not invariant.*

- *The von Neumann algebra of the action is a factor of type III.*

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Corollary 1. *Let G be semisimple with finite center and no compact factors. Suppose that M is a G-space with a harmonic probability measure μ which is ergodic but not invariant.*

- *The von Neumann algebra of the action is a factor of type III.*
- *If M is compact and the action is quasi-conformal (for some metric structure on M) then there exists a contracting fixed point.*

Two Conjectures

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An H-valued cocycle over the action of G on M is a mapping

$$\alpha : M \times G \to H$$

such that

$$\alpha(x, g_1 g_2) = \alpha(x g_1, g_2) \cdot \alpha(x, g_1).$$

There is a notion of equivalence (cohomology) of cocycles and the equivalence classes constitute the cohomology space $H^1((M, G); H)$. (A group if H is commutative.)

Example 1. *The Jacobian cocycle $j(x, g)$ of a quasi-invariant measure is a cocycle $j : M \times G \to \mathbf{R}^*_+$ because of the chain rule for the Radon-Nykodym derivative.*

For any G-space M with harmonic measure μ, the Jacobian cocycle of μ is induced by the canonical cocycle ℓ via the equivariant map $M \to JM$.

Conjecture 1. *Let G be a semisimple Lie group with finite center and without compact factors, and with rank ≥ 2. Let M be a G-space with ergodic probability harmonic measure μ. If $\alpha : M \times G \to GL(n)$ is a cocycle, then α is cohomologous to a cocycle which is induced by a cocycle on $(JM, J_*\mu)$*

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The conjecture is true in two very special cases. One is when the image JM is a point in $\mathcal{M}_1(B)$. In this case the harmonic measure μ is invariant and the conjecture is the Margulis-Zimmer superrigidity theorem.

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The other case is when JM is contained in the subspace of Dirac measures. Then $JM = B(G)$ and $J_*\mu = \beta$. The conjecture follows as a consequence of Zimmer's work on amenable actions.

The next conjecture is geometric and necessitates the concept of property (T) for G-spaces. Such concept is defined as a representation theoretic property of the pseudogroup of the action.

Conjecture 2. *Let G be semisimle with finite center and no compact factors. Suppose that M is a G-space with ergodic probability harmonic measure μ. If (M, G, μ) has property (T), then almost all the orbits of the action have only one end.*

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If G has property T, then the action of G on a one point space $M = \{\mathrm{pt.}\}$ satisfies the hypothesis and the conclusion of the theorem.