Generic aspects of the geometry of leaves of foliations

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One of the basic problems that anyone that has studied foliations asked are: what do the leaves of a foliation look like? The results presented below attempt to address this general question.

A property that the leaves of a foliated space may or may not have is called *generic* if the collection of all leaves having said property is a residual subset.

1 Geometry of leaves

A leaf of a compact foliated space has a well defined quasi-isometry type and it is a natural question to ask which quasi-isometry types of (intrinsic) metric spaces can appear as leaves of foliated spaces. There are two more or less related concepts of quasi-isometry. The first one is that used in Riemannian geometry, namely, two (Lipschitz) manifolds are quasi-isometric if there is a Lipschitz homeomorphism $f : X \to Y$. The more general concept is that in which two metric spaces X, Y are coarsely quasi-isometric if there is a mapping $f : X \to Y$ such that

1. there are constants $K \geq 1$, $A \geq 0$ so that

$$(1/K)d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + A,$$

for all $x_1, x_2 \in X$, and

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2. $f(X)$ is B-dense in Y, for some constant $B \geq 0$.

This more general concept is also used in foliation theory when discussing orbits of finitely generated pseudogroups endowed with the word metric given by a finite generating system. It is well known and elementary that an orbit of the holonomy pseudogroup of a compact foliated space is coarsely quasi-isometric to the corresponding leaf.

If G is a finitely generated group, then a choice of generating system endows the Cayley graph of G with a right invariant metric. If H is a subgroup of G, not necessarily normal, then the coset space G/H is a metric space with the induced distance. Such spaces G/H are called *discrete homogeneous spaces*. With this definition the problem of coarse quasi-isometry appearance of the leaves of a foliated space has the following answer.

Theorem 1.1. *Let $(X, \mathcal{F})$ be a compact foliated space. Then there exists a finitely generated group G such that every leaf of X is coarsely quasi-isometric to a discrete homogeneous space of G.*

Conversely, if G/H is a discrete homogeneous spaces, then there is a compact foliated space with a leaf coarsely quasi-isometric to G/H.

This is rather elementary and its main application is as technical tool in some of the results presented below. Furthermore, a result attributed to Gromov states that a Riemannian manifold is quasi-isometric to a leaf of a compact foliated space if and only it is complete and of bounded geometry. (I don't know a reference for a proof; a statement is given in Cass [5].) Whether a complete Riemannian manifold of bounded geometry is isometric to a leaf of a compact foliated space is another problem which will be discussed in the talk of J. A. Álvarez López at this conference.

The question also arises as to how many different quasi-isometry types can occur among the leaves of a given foliated space. A study of the relation "being quasi-isometric" among the leaves of a foliated space shows that its equivalence classes are Baire sets, and so basic topological dynamics provides the following answer.

Theorem 1.2. *Let $(X, \mathcal{F})$ be a transitive foliated space, either compact or with all the leaves of uniformly bounded geometry. Then either there are uncountable many (coarse) quasi-isometry types of leaves, or else there exists a residual set of leaves which are all (coarsely) quasi-isometric.*

(A foliated space is said to be *transitive* if it has a dense leaf; it is said to be *minimal* if every one of its leaves is dense.)

A further study of the equivalence relation "being quasi-isometric" sheds more light on the problem. It happens that the second possibility of the previous statement is rather restrictive.

Theorem 1.3. *Let $(X, \mathcal{F})$ be a minimal foliated space, either compact or with all the leaves of uniformly bounded geometry. Then there is a residual set of quasi-isometric leaves if and only if there is a residual set of quasi-symmetric leaves.*

Roughly speaking, a metric space is *(coarsely) quasi-symmetric* if it has sufficiently many (coarse) quasi-isometries of uniformly bounded distortion. Examples of quasi-symmetric spaces are the symmetric spaces of Lie groups, and examples of coarse quasi-symmetric spaces are finitely generated groups endowed with a word metric associated to a finite generating system.

The following corollary is a basic consequence of this result.

Corollary 1.4. *Let $(X, \mathcal{F})$ be a two-dimensional compact foliated space, which has all leaves dense and which admits no invariant transverse measure. If there is a simply connected leaf, then there is a residual set of leaves quasi-isometric to the Poincaré disk.*

The geometric structure of quasi-symmetric spaces is very homogeneous, similar to that of finitely generated groups endowed with a word metric. For instance, the end-point set of a quasi-symmetric space is either empty, a singleton, a two-point space, or a Cantor set. This has the following practical consequence.

Corollary 1.5. *In a minimal, compact foliated space with more than one leaf, if there is a leaf without holonomy whose end space is not a point, two points or a Cantor set, there there are uncountable many coarsely quasi-isometric types of leaves.*

2 Quasi-isometry invariants

On a slightly different direction, another natural question to ask is what kind of quasi-isometry invariants of metric spaces are generic for the leaves of a foliated space, *i.e.*, the same on a residual set. Well-known examples of such invariants appearing in foliation theory are the order of growth and the number of ends of leaves. A multitude of examples are the asymptotic invariants described in Gromov [10].

Quasi-isometry invariants are best understood in relation to the Gromov-Hausdorff space. This space $\mathcal{G}$ has for points isometry classes $[L, x]$ of pointed metric spaces, and is endowed with a topology in which a sequence $[L_n, x_n]$ converges to $[L, x]$ is for each $R > 0$, the closed balls $B(x_n, R)$ in L_n converge to the closed ball $B(x, R)$ in L, with respect to the Gromov-Hausdorff distance. It is thus a sort of uniform convergence on compact sets for non-compact metric spaces. The Gromov-Hausdorff space does not have a foliated structure, but, given a foliated space X, there is a canonical mapping $X \to \mathcal{G}$ which sends the point $x \in X$ to the pointed metric space $[L_x, x]$, where L_x is the leaf containing x.

Theorem 2.1. *Let X be a foliated space. The canonical mapping $x \in X \mapsto [L_x, x] \in \mathcal{G}$ is continuous on the subfoliated space of X consisting of leaves without holonomy.*

(A well-known result of Epstein, Millet and Tischler [8] and of Hector[11] says that the union of leaves with no holonomy is a dense G_δ-set in X, hence residual.)

A quasi-isometry invariant can be thought of as a function on the Gromov-Hausdorff space $\mathcal{G}$ which is constant on the equivalence classes of the obvious equivalence relation $[L,x] \sim [L,y]$ on $\mathcal{G}$. It turns out that in all the known examples such function is moreover Borel measurable. Therefore, basic topological dynamics gives the result that for a transitive foliated space X and one such Borel quasi-isometry invariant J with values in a complete separable metric space, there is a residual saturated subset of X so that all the leaves in such set have the same invariant J.

Examples where this applies are the following.

Example 2.2. One application of this fact is to the order of growth of the leaves of a foliated space.

Example 2.3. A less trivial example is the number of ends. In this case, a separate argument, having to do with recurrence, is needed to obtain the full statement of the results of Ghys [9] and Cantwell and Conlon [4].

Example 2.4. An example of a slightly different nature is the spectrum of a leaf of a foliated space endowed with a metric tensor. Here the asymptotic invariant takes values in the space of closed subsets of the line. The result is that, in a minimal foliated space, all leaves without holonomy have the same spectrum. This is a theorem of S. Hurder.

A large number of quasi-isometry invariants of metric spaces amenable to study fit into homotopy functors. Let F be a functor from the category of metric spaces to a category with limits $\mathfrak{A}$. If F is continuous (in a reasonable sense), then a quasi-isometry invariant of a space X can be defined as $F^\infty(X) = \lim_K F(X \setminus K)$, where K runs over all compact subsets K of X. For example, the space of ends is related to one particular functor, namely, to π_0.

Theorem 2.5. *Let F be a continuous functor with values in the category of vector spaces. Then $F^\infty(L)$ are isomorphic for a residual set of leaves L of a given foliated space.*

3 Compactifications

Other type of quasi-isometry invariants are played by compactifications of the leaves. A relevant compactification of a leaf, from the point of view of carrying quasi-isometric information, is the Higson-Roe compactification (see Roe [15] and Dranishnikov and Ferry [7]). This compactification of a proper metric space is constructed by considering the algebra of bounded continuous functions whose gradient decays to zero at infinity (in contrast with the end-point compactification, which is related to the algebra of bounded continuous functions whose gradient is zero outside a compact subset).

Its study has two closely related aspects; one that relates to recurrence, and studies the limit sets of points in the Higson corona, and the other to asymptotic invariants.

A leaf L of a foliated space X is Higson recurrent if the limit set of each point of the Higson corona of L is dense in X.

Theorem 3.1. *Let $(X, \mathcal{F})$ be a compact foliated space. Then the limit sets of every point of the Higson corona of a leaf is a union of leaves. Furthermore, X is minimal if and only if there is a Higson recurrent leaf.*

In certain sense, the Higson corona of a metric space is to its geometry what the end-point set is to its topology; the Higson corona keeps track of asymptotic geometric properties. This can be made slightly more precise as follows.

Theorem 3.2. *Let $(X, \mathcal{F})$ be a minimal, compact foliated space. Then the Higson coronas of any two leaves without holonomy are weakly homogeneous; in particular, they have the same topological dimension.*

(That two spaces A and B are weakly homogeneous means that for any $a \in A$ and $b \in B$, every neighborhood of a contains an open subset hoemomorphic to a neighborhood of b, and viceversa.)

Regarding asymptotic invariants, there is one related to the Higson compactification, namely, the asymptotic dimension introduced in Gromov [10]. It was shown in Dranishniknov, Keesling and Uspenkij [6] that this invariant gives a rough idea of the size of the Higson corona of a proper metric space.

Theorem 3.3. *The asymptotic dimension of the leaves of a foliated space is essentially constant.*

4 Equicontinuous foliated spaces

From a different point of view, it is also natural to try to find dynamical properties of a foliated space implying that all the leaves are quasi-isometric. The example that comes to mind is the case of Riemannian foliation, for it follows from Molino's theory [14] that the holonomy covers of all the leaves are quasi-isometric via diffeomorphism. The topological analog of Riemannian foliation is a foliated space whose holonomy pseudogroup is equicontinuous (see Ghys [14] and also Kellum [12]).

The analysis of the structure of these foliated spaces is the topic of [2]. Such analysis shows that the holonomy pseudogroup of an equicontinuous foliated space has properties similar to those of a group of isometries. However, due to the very general topological structure being studied, some further requirements are needed to realize quasi-isometries between leaves. One such particular requirement is the quasi-analyticity of the holonomy pseudogroup.

Theorem 4.1. *Let $(X, \mathcal{F})$ be a compact, equicontinuous foliate space, with connected leaf space and whose holonomy pseudogroup is quasi-analytic. Then the holonomy covers of all the leaves of X are quasi-isometric.*

Disregarding the quasi-analytic condition, the following is available. The new tool needed is the concept of normal bundle to the leaves.

Theorem 4.2. *All the universal covers of an equicontinuous foliated space with connected leaf space are quasi-isometric.*

The results of [2] permit to give, in combination with the solution to the local version of Hilbert's 5th problem, a purely topological characterization of Riemannian foliations, which is accomplished in [3]. Earlier work in this direction was that of Kellum [12, 13] who studied this problem of characterizing Riemannnian pseudogrops for certain pseudogroups of uniformly Lipschitz diffeomorphisms of Riemannian manifolds, and who broguht the local Hilbert's 5th problem to the fore. (The relevance of Hilbert's 5th problem to these issues is already noticed in Ghys [14].) A recent result, proved by Tarquini [17], states that equicontinuous transversely conformal foliations are Riemannian; in the case of dense leaves, this result of Tarquini follows easily from the main theorem in [3]. Also, Sacksteder work [16] can be interpreted as giving a characterization of Riemannian pseudogroups of one-dimensional manifolds.

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